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Admissible Estimators of λ^r in the Gamma Distribution with Truncated Parameter Space

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Abstract

In this paper, we consider admissible estimation of the parameter λ^r in the gamma distribution with truncated parameter space under entropy loss function. We obtain the classes of admissible estimators. The result can be applied to estimation of parameters in the normal, lognormal, pareto, generalized gamma, generalized Laplace and other distributions.

Keywords: Admissible estimation; Truncated parameter space; Entropy loss function

1. Introduction

Let X be a random variable with the gamma distribution

$$f(x,s,\lambda)=\frac{\lambda^s}{T(S)}x^{s-1}\exp(-\lambda x), \qquad x>0 \qquad (1.1)$$

where $s>0$ a known parameter, $\lambda\in\Lambda$ and $\Lambda=(0,\lambda_0)$ or $\Lambda=(\lambda_0,\infty),\lambda_0$ is a given constant, $\lambda_0\in R_+U\{\infty\}$. The paper deals with the problem of admissible estimation of the parameter λ^r under the entropy loss function of the form

$$L(\lambda^r,d)=\frac{d}{\lambda^r}-\ln\frac{d}{\lambda^r}-1 \qquad (1.2)$$

where r is a given constant, $r\neq 0$. Admissible estimators of λ^r were obtained by Ghosh and Singh [2]. Using Karlin's method [5,7,11] Ghosh and Singh [2] proved admissibility of the estimator $(s-2)X^{-1}$ of the parameter λ. This result was generalized by Singh [10] who showed that $\frac{\Gamma(s-r)}{\Gamma(s-2r)}X^{-r}$ is an admissible estimator of λ^r under squared error loss, where r is an integer, $r<\frac{s}{2}$ (see also [6]). Examining admissibility in the exponential family Ghosh and Meeden [1] and Ralescu and Ralescu [8] have found admissible estimators of λ and λ^{-1} in the gamma distribution (1.1) in the case of truncated parameter space under squared error loss function. Kaluszka [4] generalizes the results of Zuburzycki [13], Ghosh and Singh [2], Singh [10], Ghosh and Meeden [1] and Ralescu and Ralesca [8], when the loss is a multiple of squared error.

In this paper, we present a class of admissible estimators of the parameter λ^r in the case of restriction imposed on the parameter λ when loss is entropy loss (1.2). This result can be applied to estimation of the parameters in normal, Pareto, generalized gamma and other distributions.

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2. Admissible Estimators of λ^r with Truncated Parameter Space

Let us denote $\gamma(.,.)$, $\Gamma(.,.)$ the incomplete gamma functions, *i.e.*,

$$\gamma(x, y) = \int_0^y t^{x-1} \exp(-t) dt$$

and

$$\Gamma(x, y) = \int_y^\infty t^{x-1} \exp(-t) dt, \qquad x, y > 0$$

We now give an admissible estimator of λ^r under entropy loss function (1.2), where r is an integer, $r < \frac{s}{2}$. Let X be a random variable with gamma distribution (1.1).

Theorem 2.1. Suppose that the estimator $\hat{u}$ is of the form

$$\hat{u}(X) = \frac{\gamma(s-r, \lambda_0(k+X))}{\gamma(s-2r, \lambda_0(k+X))}(X+k)^{-r} \quad 0 < \lambda < \lambda_0$$

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where $k \geq 0$ is an arbitrary constant. Then $\hat{u}$ is admissible for λ^r under entropy loss function (1.2).

Proof. Using Karlin's method we shall first prove case (i). Suppose that there exists an estimator $\tilde{u}$ which is better than $\hat{u}$. This implies that the inequality

$$\int_0^\infty \left(\frac{\tilde{u}}{\lambda^r} - \ln\frac{\tilde{u}}{\lambda^r} - 1\right) f(x, \lambda) dx \leq \int_0^\infty \left(\frac{\hat{u}}{\lambda^r} - \ln\frac{\hat{u}}{\lambda^r} - 1\right) f(x, \lambda) dx$$

holds for all $\lambda \in \Lambda$ with strict inequality for some λ. After some calculations we get

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field codes.

Integrating both sides of (2.1) with respect to the improper prior $\xi(\lambda) = \lambda^{-r-1}\exp(-k\lambda); \lambda \in (0, \lambda_0)$, we have

$$\begin{aligned} &\int_b^{\lambda_0}\int_0^\infty \left(\frac{\tilde{u}}{\hat{u}} - \ln\frac{\tilde{u}}{\hat{u}} - 1\right) f(x,\lambda)\xi(\lambda) dx d\lambda \\ &\leq \int_b^\infty\int_0^\infty \left(\frac{\tilde{u}}{\hat{u}} - \frac{\tilde{u}}{\lambda^r} + \frac{\hat{u}}{\lambda^r} - 1\right) f(x,\lambda)\xi(\lambda) dx d\lambda \end{aligned} \tag{2.2}$$

Now by interchanging the order of integration in the right hand side of (2.2) we have

$$\begin{aligned} &\int_0^\infty \frac{\tilde{u}}{\hat{u}\Gamma(s)} x^{s-1} \int_b^{\lambda_0} \lambda^{s-r-1} e^{-(x+k)\lambda} d\lambda dx \\ &- \int_0^\infty \frac{\tilde{u}}{\Gamma(s)} x^{s-1} \int_b^{\lambda_0} \lambda^{s-2r-1} e^{-(x+k)\lambda} d\lambda dx \\ &+ \int_0^\infty \frac{\hat{u}}{\Gamma(s)} x^{s-1} \int_b^{\lambda_0} \lambda^{s-2r-1} e^{-(x+k)\lambda} d\lambda dx \\ &- \int_0^\infty \frac{1}{\Gamma(s)} x^{s-1} \int_b^{\lambda_0} \lambda^{s-r-1} e^{-(x+k)\lambda} d\lambda dx \end{aligned} \tag{2.3}$$

Now by substituting for $\hat{u}$ in (2.3) we have

$$\begin{aligned} &\int_0^\infty \frac{\tilde{u}\gamma(s-2r, \lambda_0(k+x))}{\gamma(s-r, \lambda_0(k+x))\Gamma(s)} (x+k)^r x^{s-1} \\ &\qquad \int_b^{\lambda_0} \lambda^{s-r-1} e^{-(x+k)\lambda} d\lambda dx \\ &- \int_0^\infty \frac{\tilde{u}}{\Gamma(s)} x^{s-1} \int_b^{\lambda_0} \lambda^{s-2r-1} e^{-(x+k)\lambda} d\lambda dx \\ &+ \int_0^\infty \frac{\gamma(s-r, \lambda_0(k+x))}{\gamma(s-2r, \lambda_0(k+x))\Gamma(s)} (x+k)^{-r} x^{s-1} \\ &\qquad \int_b^{\lambda_0} \lambda^{s-2r-1} e^{-(x+k)\lambda} d\lambda dx \\ &- \int_0^\infty \frac{x^{s-1}}{\Gamma(s)} \int_b^{\lambda_0} \lambda^{s-r-1} e^{-(x+k)\lambda} d\lambda dx \end{aligned} \tag{2.4}$$

Now, as $b \to 0$, $\int_b^{\lambda_0} \lambda^{s-r-1} e^{-(x+k)\lambda} d\lambda \to (k+x)^{r-s} \gamma(s-r, \lambda_0(k+x))$ and $\int_b^{\lambda_0} \lambda^{s-2r-1} e^{-(x+k)\lambda} d\lambda \to (k+x)^{2r-s} \gamma(s-2r, \lambda_0(k+x))$, and hence (2.4) tends to zero for $\lambda \in (0, \lambda_0)$.

For the case (ii) when $\lambda \in (\lambda_0, \infty)$, by integrating both sides of (2.1) with respect to the improper $\xi(\lambda) = \lambda^{-2r-1}\exp(-k\lambda); \lambda \in (\lambda_0, \infty)$, we obtain

$$\begin{aligned} &\int_{\lambda_0}^b \int_0^\infty \left(\frac{\tilde{u}}{\hat{u}} - \ln\frac{\tilde{u}}{\hat{u}} - 1\right) f(x,\lambda)\xi(\lambda) dx d\lambda \\ &\leq \int_{\lambda_0}^b \int_0^\infty \left(\frac{\tilde{u}}{\hat{u}} - \frac{\tilde{u}}{\lambda^r} + \frac{\tilde{u}}{\lambda^r} - 1\right) f(x,\lambda)\xi(\lambda) dx d\lambda. \end{aligned} \tag{2.5}$$

After interchanging the order of integration and substituting for $\hat{u}$, the right hand side of (2.5) becomes

$$\int_0^\infty \frac{\tilde{u}\Gamma\left(s-2r,\lambda_0\left(k+x\right)\right)}{\Gamma\left(s-r,\lambda_0\left(k+x\right)\right)\Gamma\left(s\right)}\left(x+k\right)^r x^{s-1} \int_{\lambda_0}^b \lambda^{s-r-1}e^{-(x+k)\lambda}d\lambda dx$$
$$-\int_0^\infty \frac{\tilde{u}x^{s-1}}{\Gamma(S)}\int_{\lambda_0}^b \lambda^{s-2r-1}e^{-(x+sk)\lambda}d\lambda dx \qquad (2.6)$$
$$+\int_0^\infty \frac{\Gamma\left(s-r,\lambda_0\left(k+x\right)\right)}{\Gamma\left(s-2r,\lambda_0\left(k+x\right)\right)\Gamma\left(s\right)}\left(x+k\right)^{-r} x^{s-1}\lambda_{\lambda_0}^b \int \lambda^{s-2r-1}e^{-(x+k)\lambda}d\lambda dx$$
$$-\int_0^\infty x^{s-1}\int_{\lambda_0}^b \lambda^{s-r-1}e^{-(x+k)\lambda}d\lambda dx.$$

Now, as $b\to\infty$, $\int_{\lambda_0}^b \lambda^{s-r-1}e^{-(x+k)\lambda}d\lambda \to \left(k+x\right)^{r-s}$ $\Gamma\left(s-r,\lambda_0\left(k+x\right)\right)$ and $\int_{\lambda_0}^b \lambda^{s-2r-1}e^{-(x+k)\lambda}d\lambda \to$ $\left(k+x\right)^{2r-s}\Gamma\left(s-2r,\lambda_0\left(k+x\right)\right)$, and hence (2.6) goes to zero for $\lambda\in\left(\lambda_0,\infty\right)$. So in both cases

$$0\le\int_0^\infty\left(\frac{\tilde{u}}{\hat{u}}-\ln\frac{\tilde{u}}{\hat{u}}-1\right)f\left(x,\lambda\right)dx\le 0$$

and this implies that

$$\int_0^\infty\left(\frac{\tilde{u}}{\hat{u}}-\ln\frac{\tilde{u}}{\hat{u}}-1\right)f\left(x,\lambda\right)dx=0$$

i.e., $\tilde{u}=\hat{u}$ a.e., proving admissibility of $\hat{u}\left(x\right)$.

3. Applications to Other Distributions

Using adequate transformations, from Theorem 2.1, we can obtain admissible estimators of parameters of some well-known distributions. For other distributions, the reader can refer to [9].

Example 3.1. (The pareto Distribution) The pareto distribution with density given by

$$f\left(y,\lambda,a\right)=\lambda a^\lambda y^{-\lambda-1} \quad a<y<\infty;a,\lambda>0 \qquad (3.1)$$

will be considered. Suppose that $Y_1,\ldots,Y_n$ is a random sample from (3.1) with given a and note that the random variable $Y=\sum_{i=1}^n \ln Y_i$ has the gamma distribution (1.1) for $s=n$. Hence applying theorem 2.1, we get admissible estimator of $\lambda^r,\lambda\in\left(0,\infty\right)$ of the form

$$\frac{\Gamma\left(n-r\right)}{\Gamma\left(n-2r\right)}\left(-n\ln a+\sum_{i=1}^n \ln Y_i\right)^{-r} \qquad r<\frac{n}{2}$$

Example 3.2. (Generalized Laplace Distribution) Let Y have the density

$$f\left(y,b,k\right)=\frac{k}{2b\Gamma\left(\frac{1}{k}\right)}\exp\left(-\frac{|y|^k}{b^k}\right), \qquad (3.2)$$
$$y\in R,b,k>0$$

where k is a given parameter. Let $Y_1,\ldots,Y_n$ be a random sample from (3.2) and put

$$\delta\left(Y_1,\ldots,Y_n\right)=\frac{\Gamma\left(\frac{n+r}{k}\right)}{\Gamma\left(\frac{n+2r}{k}\right)}\left(\sum_{i=1}^n|Y_i|^k\right)^{\frac{r}{k}}$$

where $r>-\frac{n}{2}$. The above estimator is admissible for b^r under entropy loss function. Hence, the estimators of the power of the scale parameter in the normal and Laplace distributions can be found.

Example 3.3. Suppose that $Y_1,\ldots,Y_n$ is a random sample from the generalized gamma distribution with the density function given by

$$f\left(y,p,\lambda,\alpha\right)=\frac{|\alpha|}{\Gamma\left(\frac{p}{\alpha}\right)}\lambda^{\frac{p}{\alpha}}y^{p-1}\exp\left(-\lambda y^\alpha\right),$$
$$0<y<\infty,p\alpha>0$$

where p,α are given parameters. Observe that the random variable $Y=\sum_{i=1}^n Y_i^\alpha$ has the gamma distribution (1.1) for $s=\frac{np}{\alpha}$, so one can obtain estimators of parameters in Maxwell, Weibull, Rayleigh, and other distributions.

In a similar way, we get admissible estimators of parameters in log-normal, special cases of beta, Burr distribution [3,12] and a few others.

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