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# Scheduled Review Methods for Controllable State Variables

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## Abstract

In many real systems in which a state variable should be controlled for being in appropriate range, the length of control (review) intervals is taken to be constant. In such systems, when the cost of reviews and out-of-range values of the state variable are considerable, this method may not be optimal. In this paper we let the length of review intervals to be variable during each operating cycle and construct the related mathematical cost model. Then two scheduled review methods, called $\underline{U}_2$ and $\underline{U}_3$, are introduced and the relative annual system costs are analyzed. The model is developed for the case of negative exponential variate as the time between successive consumption points. It is shown that the new methods results a significant reduction in the expected annual cost of the system.

**Keywords:** State variables; Inventory systems; Random variates; Review methods

## 1. Introduction

In many real systems there exist critical state variables whose values must be kept within a predefined range. If they cross the range boundaries, a cost will be incurred. The nature of the state variables in such systems is stochastic so that it is not known (or computable) when it will reach its boundaries. Usually, in order to control a state variable in such system, the values of the variable are reviewed either continuously or periodically. Certainly, when reviews incur cost and take time they have to be performed discontinuously.

At a review time, if it is observed that the variable has crossed the borders, proper action is taken to adjust the value of the variable. Production and inventory systems, security systems inspection, soil moisture in agricultural activities and some of medical systems are well known examples of such systems.

A system whose annual cost model is considered in this paper consists of a state variable *V,* which, due to stochastic consumption, is always decreasing (except when its value is adjusted). It is assumed that the upper and lower boundaries are positive real number $\omega$ and ν, respectively. When, by a review, the value of *V* is observed to be equal to or less than ν it is increased up to the level ω. The review and increment cost is fixed at A. In some systems, such as inventory systems, negative net values of V cause both constant and time dependent costs.

This kind of cost is called shortage or backlog cost. We assume that each negative unit of V during a time of length t incurs $\pi + \hat{\pi}$ t unit of cost. Of course holding positive values for V adds up to the system's cost. Assume this cost is h for a positive unit of V for one unit of time.

It is obvious that the purpose of making a review is

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to check whether V has reached or crossed ν. If it has not, the review would not have any effect except its cost. In most of the so far proposed review methods the inter-review times (periods) are taken to be equal and constant, see for example [1-3]. In order to decrease the overall shortage cost, Teunter *et al.* [6] suggested emergency reviews in addition to the regular reviews. In fact, the decreasing nature of V implies that the subsequent review intervals need not to be of the same length as the preceding ones. Salehi Fathabadi [4] introduced an inventory control policy in which the lengths of review periods are computed as a function of inventory level. Within this policy the total annual system cost will considerably decrease when shortage costs are high.

In this paper, first we construct a general annual cost model for variable review intervals. Three reviewing methods are described and applied to the general model. Then methods' efficiency, when the time length between consumption points is negative exponential, are compared.

Through the paper we assume that the consumption time is stochastic and occurs in single unit; the intervals between successive consumption events are independent and identically distributed.

## 2. The General Cost Model

Let $X_t$ be the consumption on V during a time interval of length t with probability function p(x, t) and mean μt, where μ is the expected consuming rate. Let the time origin be defined as the instant of increasing the state variable up to ω. Also suppose $U_r$, r = 1, 2, 3,… are the review times measured from the time origin and $U_0 = 0$.

Now let Z be the time taken by the state variable to reach the level ν, starting from the time origin. If the p.d.f of Z is denoted by $f(z)$ and its cumulative distribution function by F(z), then

$$F(z) = \sum_{x=m}^{\infty} p(x;z) = 1 - \sum_{x=0}^{m-1} p(x;z) \tag{2.1}$$

and $f(z) = dF(z)/dz$, where m=ω−ν.

Suppose the value of V is observed to be ν or less at a review time $U_r$, so that $U_{r-1}<z\le U_r$. The cycle length in this case is $U_r$. Therefore, if the number of reviews during a cycle is denoted by R, its expected value will be

$$E(R) = \sum_{r=1}^{\infty} r[F(U_r) - F(U_{r-1}) ,$$

and the expected cycle length is

$$E[U_R] = \sum_{r=1}^{\infty} U_r[F(U_r) - F(U_{r-1})] . \tag{2.2}$$

At any time t during a cycle, $0 < t \le U_r$, the expected net value of V is ω - μt and the expected shortage is 0 for $t \le z$ and

$$\sum_{x=v}^{\infty} (x-v)p(x;t-z)$$

for t > z. Define $V^+ = \text{Max.}\ \{V, 0\}$ and $V^- = \text{Max}\ \{-V, 0\}$. Since $V^+ = V + V^-$ the expected cost of holding positive V, subjects to $U_{r-1} < z \le U_r$ is

$$\begin{aligned} C_1(U_r) = h[&\int_o^{u_r} (\omega - \mu t)dt \\ &+ \int_o^{u_r} \sum_{x=v}^{\infty} (x-v)p(x;t-z)dt] \end{aligned} \tag{2.3}$$

Assuming $\pi + \hat{\pi}$ t as the cost of one unit of $V^-$ for a time interval of length t, the conditional expected cost of shortage in a cycle is

$$\begin{aligned} C_2(U_r) = \pi &\sum_{x=v}^{\infty} (x-v)p(x,U_r - z) \\ &+ \hat{\pi} \int_o^{u_r} \sum_{x=v}^{\infty} (x-v)p(x;t-z)dt. \end{aligned} \tag{2.4}$$

Therefore the expected total cost of operating the system per cycle. $E[C_R]$, is

$$\begin{aligned} E[C_R] &= \sum_{r=1}^{\infty} \int_{U_{r-1}}^{U_r} [A + rJ + C_1(U_r) \\ &\qquad + C_2(U_r)]f(z)dz \\ &= A + JE[R] \\ &\qquad + \sum_{r=1}^{\infty} \int_{U_{r-1}}^{U_r} [C_1(U_r) + C_2(U_r)]f(z)dz . \end{aligned} \tag{2.5}$$

where J is the cost of one review. We use $E[C_R]/E[U_R]$ as a measure of the expected total annual cost. Let

$$(h+\hat{\pi})\int_z^{U_r} \sum_{x=v}^{\infty} (x-v)p(x;t-z)dt$$

$$+\pi \sum_{x=v}^{\infty} (x-v)p(x;t-z)$$

be denoted by g ($U_r$, z) and $\{U_r\}_{r=0}$ by $\underline{U}$. An easy simplification of (2.5) gives the expected annual cost of the system as:

$$K(v,\omega,\underline{U}) = \{A + JE[R] + h\omega E[U_R] - \frac{1}{2}\mu h E[U_R^2] + \sum_{r=1}^{\infty}\int_{U_{r-1}}^{U_r} g(U_r,z)f(z)dz\} / E[U_R] \qquad (2.6)$$

In general case we assume the consumption distribution provides for existence of $\sum_{r=0}^{\infty} F(U_r)$ and $\sum_{r=0}^{\infty} rF(U_r)$ where F' (.) = 1 − F (.). It is easy to see that

$$E[R] = \sum_{r=0}^{\infty} F'(U_r) - \lim_{n\to\infty} nF'(U_n) = \sum_{r=0}^{\infty} F'(U_r), \qquad (2.7)$$

According to the above assumption E[R] exists.

Now consider that

$$g(U_r,z) = (h+\hat{\pi})\int_z^{U_r} \mu(t-z)dt + \pi\mu(U_r - z) = 1/2(h+\hat{\pi})\mu\beta^2 + \pi\mu\beta$$

where $\beta = U_r - z$. For non-increasing series of $\{U_r - U_{r-1}\}_{r=1}$,

$$\sum_{r=1}^{\infty}\int_{U_{r-1}}^{U_r} g(U_r,z)f(z)dz \le \mu\sum_{r=1}^{\infty}[1/2(h+\hat{\pi})(U_r - U_{r-1})^2 + \pi(U_r - U_{r-1})]\int_{U_r-1}^{U_r} f(z)dz \le \mu[1/2(h+\hat{\pi})U_1 + \pi]U_1$$

Thus $\sum_{r=1}^{\infty}\int_{U_{r-1}}^{U_r} g(U_r,z)f(z)dz$ is finite. Also if only the first n terms of this summation are summed up, the error will be

$$e_n = \sum_{r=n+1}^{\infty}\int_{U_{r-1}}^{U_r} g(U_r,z)f(z)dz \le \mu[1/2(h+\hat{\pi})(U_{n+1} - U_n)^2 + \pi(U_{n+1} - U_n)]F'(U_n) \qquad (2.8)$$

Since $F'(U_n) \to 0$ as $n\to\infty$ and $(U_{n+1} - U_n) \le U_1$, $e_n$ tends to 0 as $n\to\infty$. Hence $e_n$ can be made as small as desired.

## 3. Review Methods

In the classical treatment of the discrete review policy [2], as stated earlier, the length of all review intervals are taken as equal. In many cases such policy represents an easy schedule and implementation from a practical point of view. But the increasing hazard rate function, $f(z)/[1-F(z)]$, of Z provides for succeeding review intervals shorter than the preceding ones. We consider two other simple methods for generating review times. In the first method the length of the first period is different from the later periods. In the second one the first two periods differ from the others. The classical review method is also included in the analyzing of the cost function and comparison.

### *3.1. Periodic Review Method, $\underline{U}_1$*

In this method review times are generated as:

$$U_r = rT, \quad r = 1,2,3,\ldots \qquad (3.1)$$

Let $\underline{U}_1$ denotes the set of times generated by this method. We have

$$E[U_R] = T\sum_{r=0}^{\infty} r[F'(U_{r-1}) - F'(U_r)] = TE[R] = T\sum_{r=0}^{\infty} F'(U_r), \qquad (3.2)$$

and

$$E[U_R^2] = T^2\sum_{r=1}^{\infty} r^2[F'(U_{r-1}) - F'(U_r)] = T^2[2\sum_{r=0}^{\infty} rF'(U_r) + \sum_{r=0}^{\infty} F'(U_r)], \qquad (3.3)$$

since, the existence of $\sum_{r=0}^{\infty} rF(U_r)$ implies that $n^2F'$

$(U_n) \to 0$ as $n \to \infty$. Hence E $[U_R]$ and E $[U_R^2]$ exist and the cost function can be calculated to any level of accuracy.

### *3.2. One Period Scheduled Review Method, $\underline{U}_2$*

In this method, apart from the first review in a cycle, the reviews are periodic. Let the length of the first review period be denoted by $T_1$ and the length of the subsequent periods be denoted by T. Thus review times, $\underline{U}_2$, are generated by:

$$U_r = T_1 + (r-1)T\,, r = 1,2,\ldots \tag{3.4}$$

It is obvious that $T_1$ is not less then T, so that $\{U_r - U_{r-1}\}_{r=1}$ is non-increasing. Using the convergence assumption of $\sum_{r=0}^{\infty} F(U_r)$ and $\sum_{r=0}^{\infty} rF(U_r)$ we have

$$\begin{aligned} E[U_R] &= \sum_{r=1}^{\infty}[T_1 + (r-1)T][F'(U_{r-1}) - F'(U_r)] \\ &= T_1 + T\sum_{r=1}^{\infty} F'(U_r) \end{aligned} \tag{3.5}$$

This is convergent to a finite time length. Also it is easy to see that.

$$\begin{aligned} E[U_R^2] &= T_1^2 + 2T^2\sum_{r=0}^{\infty} rF'(U_r) \\ &+ T(2T_1 - T)\sum_{r=1}^{\infty} F'(U_r) \end{aligned} \tag{3.6}$$

### *3.3. Two Periods Scheduled Review Method, $\underline{U}_3$*

In this method the first two review intervals are different in length from the subsequent intervals. The review times, $\underline{U}_3$, are scheduled as:

$$\begin{aligned} &U_1 = T_1, \\ &U_r = T_1 + T_2 + (r-2)T\,, \quad r = 2,3,4,\ldots \end{aligned} \tag{3.7}$$

Obviously $T_1 \ge T_2 \ge T$ and, within this method, $\{U_r - U_{r-1}\}_{r-1}$ is also non-increasing. In this method, with a simple formulation we get

$$E[U_R] = T_1 + T_2F'(T_1) + T\sum_{r=2}^{\infty} F'(U_r) \tag{3.8}$$

and

$$\begin{aligned} E[U_R^2] &= T_1^2 + T_2(2T_1 + T_2)F'(T_1) \\ &+ T(2T_1 + 2T_2 - 3T)\sum_{r=2}^{\infty} F'(U_r) \\ &+ 2T^2\sum_{r=2}^{\infty} rF'(U_r) \end{aligned} \tag{3.9}$$

## 4. Negative Exponential Consumption Intervals

We suppose that the intervals between successive occurrence of consumption have negative exponential density function. That is,

$$g(t) = \lambda e^{-\lambda t}, \quad t > 0\,. \tag{4.1}$$

This gives

$$p(x,t) = (\lambda t)^x \lambda e^{-\lambda t} / x\,!, \quad x = 0,1,2,\ldots \tag{4.2}$$

and

$$f(z) = \frac{\lambda^m z^{m-1} e^{-\lambda z}}{(m-1)!}, \tag{4.3}$$

This shows that Z is a Gamma variate.
Since, for this probability function,

$$\begin{aligned} &\lim_{r\to\infty} F'(U_r + 1)/F'(U_r) \\ &= \lim_{r\to\infty}(r+1)F'(U_{r+1})/[rF'(U_r)] \\ &= e^{-\lambda t} \end{aligned} \tag{4.4}$$

for all the three methods, $\sum_{r=0}^{\infty} F(U_r)$ and $\sum_{r=0}^{\infty} rF(U_r)$ exist, and therefore the assumptions about these terms are valid.

### *4.1. Numerical Evaluation of Optimal Decision Variables*

For numerical determination of optimal decision variables, ν, ω, $T_1$, $T_2$ and T, either a numerical method capable of handling non-linear functions, mixed integer and real variables has to be exploited, or an iterative procedure is applied. To apply the latter approach an upper bound on m has to be established.

First Euler-Maclaur summation formula, is used to find exact or nearly exact values of $E[U_R]$ and $E[U^2_R]$ for $\underline{U}_1$, $\underline{U}_2$ and $\underline{U}_3$ review times. The formula is

$$\sum_{r=0}^{n}\Phi(r)=\int_0^n \Phi(x)dx+\frac{1}{2}(\Phi(n)+\Phi(0))$$
$$+(\Phi^{(1)}(n)-\Phi^{(1)}(0))/12$$
$$-(\Phi^{(3)}(n)-\Phi^{(3)}(0))/720+\cdots \quad (4.5)$$
$$+B_{2k}(\Phi^{(2k-1)}(n)$$
$$-\Phi^{(2k-1)}(0))/2k!+B_k .$$

where $B_k$'s are Bernoulli numbers and $\Phi^{(j)}(.)$ is the $j^{th}$ derivative of $\Phi(.)$ and $R_k$ is the remainder which is smaller, in absolute value, than the first neglected term.

Denoting (4.3) by $f_m(z)$, taking $\sigma_1 = T_1 - T$, $\sigma_2 = T_2 + T_1 - 2T$ and using (4.5) we get the following results for the three review methods:

**(i)** – For $\underline{U}_1$ and m>3:

$$E[R]=m/(\lambda T)+1/2 \quad (4.6)$$

$$E[U_R]=m/\lambda+T/2 \quad (4.7)$$

$$E[U_R^2]=m(m+1)\lambda^2+mT/\lambda+T^3/3 \quad (4.8)$$

**(ii)** – For $\underline{U}_2$:

$$E[R]=1+mF'_{m+1}(\sigma_1)/(\lambda T)$$
$$-(T_1/T-1/2)F'_m(\sigma_1)+Tf_m(\sigma_1)/12 \quad (4.9)$$

$$E[U_R]=T_1+mF'_{m+1}(\sigma_1)/\lambda-\sigma_1 F'_m(\sigma_1)$$
$$-TF'_m(\sigma_1)/2+T^2 f_m(\sigma_1)/12 \quad (4.10)$$

$$E[U_R^2]=m(m+1)F'_{m+2}(\sigma_1)/\lambda^2$$
$$+mTF'_{m+1}(\sigma_1)/\lambda+T_1^2$$
$$-(T_1^2-T^2/3)F'_m(\sigma_1) \quad (4.11)$$
$$+T^2(T_1+\sigma_1)f_m(\sigma_1)/12$$

**(iii)** – For $\underline{U}_3$:

$$E[R]=1+mF'_{m+1}(\sigma_2)/(\lambda T)$$
$$-(\sigma_2/T+1/2)F'_m(\sigma_2)+F'_m(\sigma_2) \quad (4.12)$$
$$-F'_m(\sigma+T)Tf_m(\sigma_2)/12$$

$$E[U_R]=T_1+T_2F'_m(T_1)$$
$$-(T/2+\sigma_2)F'_m(\sigma_2)+F'_{m+1}(\sigma_2)/\lambda \quad (4.13)$$
$$-TF'_m(\sigma_2+T)T^2 f_m(\sigma_2)/12$$

$$E[U_R^2]=T_1^2+T_2(2T_1+T_2)F'_m(T_1)$$
$$-(2T^2/3+\sigma_2^2)F'_m(\sigma_2)$$
$$-(2T\sigma_2+3T^2)F'_m(T+\sigma_2)$$
$$+T^2(T+2\sigma_2)f_m(\sigma_2) \quad (4.14)$$
$$+m(m+1)F'_{m+2}(\sigma_2)/\lambda^2$$
$$+mTF'_{m+1}(\sigma_2)/\lambda$$

It should be reminded that the error in the use of (4.9) to (4.14) is very small for moderately large m. In order to get an upper bound for m, the value of $\sum_{r=1}^{\infty}\int_{U_{r-1}}^{U_r} g(U_r,z)f(z)dz$ must be found for at least large values of m.

**Theorem 1.** In the review methods $\underline{U}_1$, $\underline{U}_2$ and $\underline{U}_3$

$$\lim_{r\to\infty}\sum_{r=1}^{\infty}\int_{U_{r-1}}^{U_r} g(U_r,z)f_m(z)dz=G(T)/T \quad (4.15)$$

where

$$G(T)=\int_0^T \{\frac{1}{2}\lambda y[2\pi+(h+\hat{\pi})y]\bar{P}(v-1;y)$$
$$+v(v+1)(h+\hat{\pi})\bar{P}(v+1;y)/(2\lambda)$$
$$-v[\pi+(h+\hat{\pi})y]P(v;y)\}dy .$$

and $P$(K, y) is the complementary cumulative distribution of poison variate.

**Proof.** The proof is for the three review method separately. First consider that by using

$$\sum_{x=v}^{\infty}(x-v)p(x;t-z)=\lambda(t-z)\bar{P}(v-1;t-z)$$
$$-v\bar{P}(v;t-z)$$

and

$$\int_0^T t^n P(v;t)dt=T^{n+1}P(v;T)/(n+1)$$
$$-(v+n)!P(n+v+1;T)/[\lambda^{n+1}(n+1)(v-1)!]$$

$g$ ($U_r$, z) can be written as:

$$g_0(\beta) = \frac{1}{2}\lambda\beta[2\pi + (h+\hat{\pi})\beta]\bar{P}(v-1;\beta) + v(v+1)(h+\hat{\pi})\bar{P}(v-1;\beta)/(2\lambda) - v[n+(h+\hat{\pi})\beta]\bar{P}(v;\beta) \quad (4.16)$$

**(a)** $U_r = rT$, r = 1, 2, 3. Let n be an integer satisfying $n-1 < (m-1)/\lambda T \le n$ for m > 1, and n = 1 for m = 1. Let $\alpha = \text{Min.}\{f_m(U_{n-1}), f_m(U_n)\}$. Since $g_0(\beta) \ge 0$,

$$\sum_{r=1}^{\infty}\int_{U_{r-1}}^{U_r} g_0(\beta) f_m(z)dz \ge G(T)[\sum_{r=1}^{\infty} f_m(U_{r-1}) - f_m(U_{n-1}) + f_m(U_n) + \alpha] \quad (4.17)$$

Using (4.5), for m>1 we get

$$\sum_{r=1}^{\infty} f_m(U_r) = 1/T \ ,$$

Thus

$$\sum_{r=1}^{\infty}\int_{U_{r-1}}^{U_r} g_0(\beta) f_m(z)dz \ge G(T)/T - G(T)[f_m(U_{n-1}) + f_m(U_n) - \alpha] \quad (4.18)$$

Also

$$\sum_{r=1}^{\infty}\int_{U_{r-1}}^{U_r} g_0(\beta) f_m(dz) \le G(T)[\sum_{r=1}^{\infty} f_m(U_r) + f_m(\frac{m-1}{\lambda})] = G(T)/T + G(T) f_m(\frac{m-1}{\lambda}) \quad (4.19)$$

Since $f_m(U_n)$, $f_m(U_{n-1})$ and $f_m((m-1)/\lambda)$ tends to zero as $m\to\infty$, the proof in this case is completed.

**(b)** $U_r = T_1 + (r-1)T$, r = 1, 2, 3…. In this case suppose $m > \lambda T_1 + 1$ and n satisfies $T_1 + (n-2)T < (m-1)/\lambda \le T_1 + (n-1)T$. Then in a similar way we get

$$\sum_{r=1}^{\infty}\int_{u_{r-1}}^{u_r} g_0(\beta) f_m(z)dz \le G(T_1) f_m(T_1) + G(T)[\sum_{r=2}^{\infty} f_m(U_r) + f_m(\frac{m-1}{\lambda})].$$

Applying Euler's formula, we have

$$\sum_{r=2}^{\infty} f_m(U_r) = F'_m(\delta_1)/T - 1/2 f_m(\delta_1) - f_m(T_1) + \text{Re}\,.$$

where

$$|\,\text{Re}\,| \le \lambda T[f_{m-1}(\delta_1) - f_m(\delta_1)]\,.$$

Therefore

$$\sum_{r=1}^{\infty}\int_{u_{r-1}}^{u_r} g_0(\beta) f_m(z)dz \le \frac{G(T)}{T}F'_m(\delta_1) + G(T_1) f_m(T_1) + G(T)[f_m(\frac{m-1}{\lambda}) - 1/2 f_m(\delta_1) - f_m(T_1) + \lambda T f_{m-1}(\delta_1)/12 - \lambda T f_m(\delta_1)/12] \quad (4.20)$$

Similarly

$$\sum_{r=1}^{\infty}\int_{u_{r-1}}^{u_r} g_0(\beta) f_m(z)dz \ge \frac{G(T)}{T}F'_m(\delta_1) + G(T)[\lambda T f_m(\delta_1)/12 - \lambda T f_{m-1}(\delta_1)/12 - 1/2 f_m(\delta_1) - f_m(U_{u-1}) - f_m(U_n) + \alpha] \quad (4.21)$$

where α is defined as in case (a). Note that

$$1 - \delta_1 f_m(\frac{m-1}{\lambda}) \le F'_m(\delta_1) \le 1 \quad (4.22)$$

and lim F' $(\delta_1) = 1$ when $m\to\infty$. Canceling all terms inside the square brackets in (4.20) and (4.21), when $m\to\infty$, proves Theorem in this case.

**(c)** $U_1 = T_1$, $U_r = T_1 + T_2 + (r-2)T$, r = 1, 2, 3. In this case, by taking $m > \lambda(T_1 + T_2) + 1$ and n satisfying $T_1 + T_2 + (n-3)T < (m-1)/\lambda \le T_1 + T_2 + (n-2)T$ the proof is similar to case (b).

To establish the upper bound on m, first consider the $\underline{U}_1$ method. Let for a precision factor, ε, $M_1$ be such an integer that if $m > M_1$ then the absolute difference between $\sum_{r=1}^{\infty}\int_{u_{r-1}}^{u_r} g_0(\beta) f_m(z)dz$ and G (T)/T is less than ε. Let M = Max. $\{3, M_1\}$. For m > M the cost function can, with the accuracy level ε, be written as

$$K(\nu,\omega,\underline{U}_1)=\frac{2\lambda A}{2m+\lambda T}+J/T+h\omega$$
$$+h\frac{m^2-m-\lambda^2T^2/3}{2m+\lambda T} \tag{4.23}$$
$$+\frac{2\lambda G(T)}{T(2m+\lambda T)}$$

By taking m as a continuous variable and solving $\partial K/\partial m=0$, we have

$$m_1^0=[2\lambda(AT+G(T))/(hT)-\lambda^2T^2/12 \\ +\lambda T/2]^{1/2}-1/2\lambda T \tag{4.24}$$

Now an upper bound on m can be taken as:

$$m_1(T)=\text{Max.}\{3,M_1,[m_1^0]+1\} \tag{4.25}$$

in which [x] is the greatest integer less than or equal to x.

In the case of $\underline{U}_2$ and $\underline{U}_3$, let $M_2$ and $M_3$ be the counterparts of $M_1$ with the corresponding T. Considering (4.9) to (4.11) in case of $U_2$, it is not difficult to see that if $m > M_2$ then

$$E[R]=E[U_R]/T-\delta_1/T \tag{4.26}$$

$$E[U_R]=m/\lambda+T/2 \tag{4.27}$$

and

$$E[U_R^2]=m(m+1)/\lambda^2+mT/\lambda+T^2/3 \tag{4.28}$$

Also using (4.12)-(4.14), we see that if $m > M_3$ then E[R], E $[U_R]$ and E $[U_R^2]$ for $\underline{U}_3$ have the same values as (4.26)-(4.28) respectively with $\delta_1$ substituted by $\delta_2$. Finally the annual cost function in case of $\underline{U}_2$ and $\underline{U}3$ methods becomes

$$K(\nu,\omega,\underline{U}_i)=\frac{2\lambda A}{2m+\lambda T}+J/T+h\omega$$
$$+h\frac{m^3-m\lambda^2T^2/3}{2m+\lambda T} \tag{4.29}$$
$$+\frac{2\lambda[G(T)-J\delta_{i-1}]}{T(2m+\lambda T)}$$

Taking the same approach as in the case of $\underline{U}_1$ method, the upper bound for m in $\underline{U}_2$ method will be

$$m_2(T_1,T)=\text{Max.}\{M_2,[m_2^0]+1\} \tag{4.30}$$

where

$$m_2^0=[2\lambda(AT+G(T)-J\delta_1)/(hT) \\ -\lambda^2T^2/12+\lambda T/2]^{1/2}-1/2\lambda T. \tag{4.31}$$

In the case of $U_3$, $m^0_3$ is the same as (4.31) with $\delta_1$ replaced by $\delta_2$ and the upper bound on m is

$$m_2(T_1,T_2,T)=\text{Max.}\{M_3,[m_3^0]+1\} \tag{4.32}$$

### *4.2. Model Optimization*

As it is seen, minimization of the annual cost function is very complicated. It is a mixed integer and real optimization problem for which no known method exists. Therefore a heuristic method has to be applied.

Being able to determine an upper bound for m we suggest the following iterative search procedure to evaluate the optimal values of the related decision variables.

1. Guess values for T, $T_1$, and $T_2$ as appropriate.
2. Guess ν.
3. Evaluate $m_1$ (T), $m_2$ ($T_1$, T) or $m_3$ ($T_1$, $T_2$, T) as appropriate.
4. Find $\omega_0$ in the interval (ν, $m_i$ + ν) which gives the lowest cost function value.
5. Fixing ω to $\omega_0$ find $\nu_0$ which gives the lowest cost function value.
6. If $\nu_0$ is different from ν, set ν = $\nu_0$ and start from step 3.
7. Using $\omega_0$ and $\nu_0$, find the related optimal value of T, $T_1$, and $T_2$ as appropriate.
8. If the predefined accuracy level on interval lengths has been reached stop otherwise start from step 3.

In general for given values of ν and ω, the annual cost function has several minima in terms of the review intervals. This creates the possibility that a normal search procedure terminates with a local solution. Figures 1, 2 and 3 illustrate this point.

Further numerical investigations suggest that K is unimodal with respect to T in both $U_2$ and $U_3$ when $T_1$ (and $T_2$) are fixed. Furthermore when T is set to its optimal value, then K is unimodal with respect to $T_1$ in both of the methods (see Figs. 4 and 5).

The above remarks suggest that if steps 7 and 8 in the search procedure are replaced by the following steps then the results are very likely to be global. Given an initial value of T1 (E[z] for example):

1. Minimize K with respect to T. Denote the best value found for T as $T_0$.
2. Set T = $T_0$ and search for the best value of $T_1$ and denote it by $T_{1,0}$.
3. Use $T_{1,0}$ as the initial value of $T_1$ and start from

step 1.

The optimal values of $T_1$ and T are found when two consecutive values of T or $T_1$ are equal.

In the investigation on review periods, it has been observed that $T_1$ is the most effective period in $\underline{U}_2$ and $\underline{U}_3$. For example, in case $\underline{U}_2$, $T_1$ causes a significant variation in K ($T_1$) when T is optimal. In contrast, for optimal $T_1$, K (T) is rather a flat function. Tables 1 and 2 demonstrate the effect of the first review periods on the annual cost function in the cases of $\underline{U}_2$ and $\underline{U}_3$.

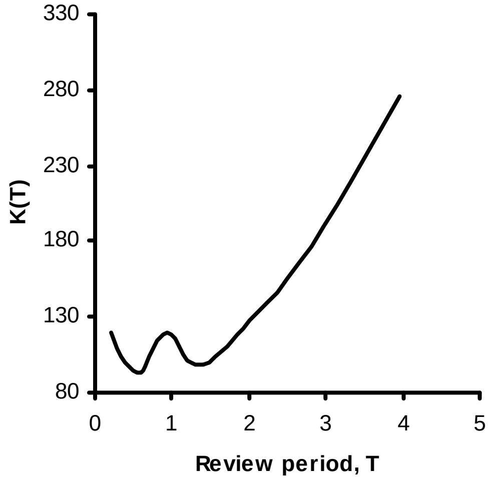


**Figure 1.** Total cost of $\underline{U}_1$ method.

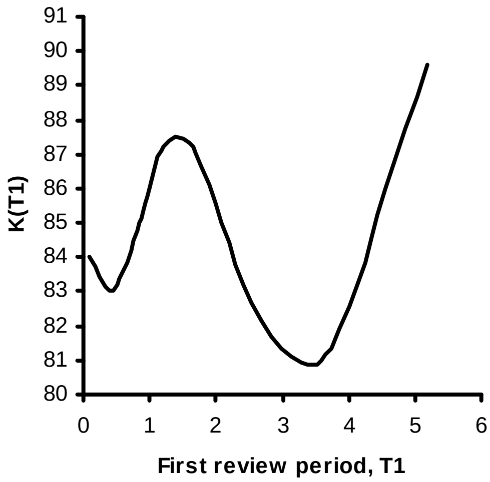


**Figure 3.** Total cost of $\underline{U}_3$ method.

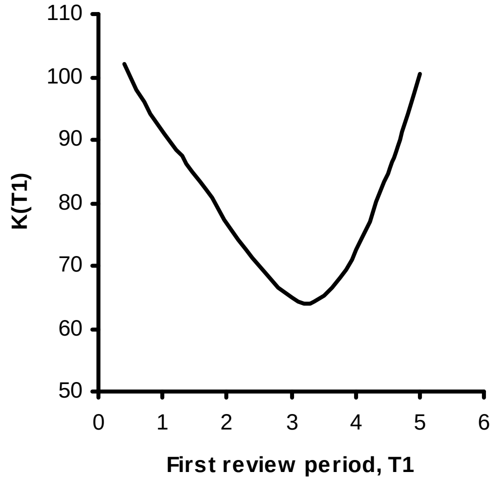


**Figure 5.** Total cost of $\underline{U}_3$ for optimal $T_2$ and $T_3$.

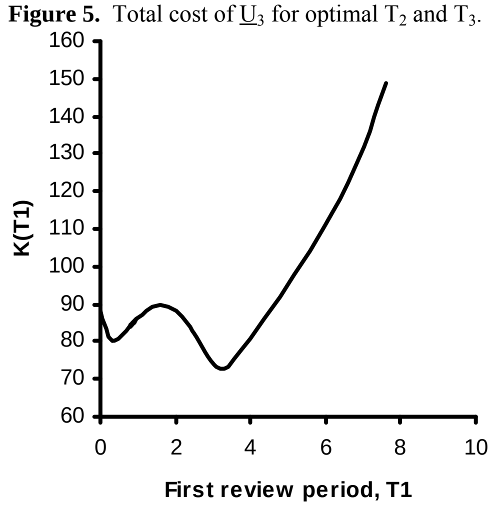


**Figure 2.** Total cost of $\underline{U}_2$ method.

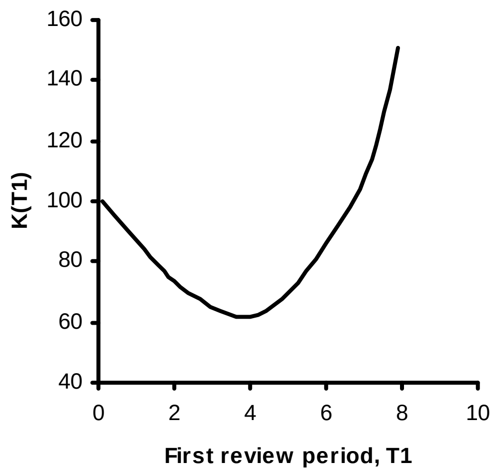


**Figure 4.** Total cost of $\underline{U}_2$ method for optimal T.

**Table 1.** Total cost of U$_2$ for different review periods

| ***T* = 0.54** | | | ***T*$_1$ = 1.69** | | |
|---|---|---|---|---|---|
| $T_1$ | $K(T_1)$ | Increase | $T$ | $K(T)$ | Increase |
| 1.0 | 126.2 | 30.1 | 0.1 | 97.05 | 0.04 |
| 1.2 | 112.7 | 16.1 | 0.2 | 97.03 | 0.01 |
| 1.4 | 102.0 | 5.2 | 0.3 | 97.02 | 0.00 |
| 1.6 | 97.4 | 0.4 | 0.4 | 97.01 | 0.00 |
| 1.69* | 97.0 | 0 | 0.54* | 97.01 | 0 |
| 1.7 | 97.0 | 0.0 | 0.7 | 97.01 | 0.00 |
| 1.9 | 98.9 | 1.9 | 0.9 | 97.01 | 0.00 |
| 2.1 | 103.3 | 6.4 | 1.1 | 97.02 | 0.01 |
| 2.3 | 109.6 | 12.9 | 1.3 | 97.03 | 0.02 |

**Table 2.** Total cost of U$_3$ for different review periods

| ***T* = 0.62, *T* = 0.55** | | | ***T*$_2$ = 1.69, *T* = 0.55** | | | ***T*$_2$ = 1.69, *T*$_2$ = 1.69, *T*$_1$ = 0.62** | | |
|---|---|---|---|---|---|---|---|---|
| $T_2$ | $K(T_2)$ | Increase | $T_1$ | $K(T_1)$ | Increase | $T$ | $K(T)$ | Increase |
| 1.0 | 124.4 | 28.2 | 0.3 | 97.02 | 0.01 | 0.2 | 97.01 | 0.00 |
| 1.2 | 112.3 | 15.7 | 0.4 | 97.01 | 0.00 | 0.3 | 97.01 | 0.00 |
| 1.4 | 102.0 | 5.1 | 0.5 | 97.01 | 0.00 | 0.4 | 97.01 | 0.00 |
| 1.6 | 97.4 | 0.4 | 0.6 | 97.01 | 0.00 | 0.5 | 97.01 | 0.00 |
| 1.69* | 97.0 | 0 | 0.62* | 97.01 | 0′ | 0.55* | 97.01 | 0 |
| 1.7 | 97.0 | 0.0 | 0.7 | 97.01 | 0.00 | 0.7 | 97.01 | 0.00 |
| 1.9 | 98.8 | 1.8 | 0.8 | 97.01 | 0.00 | 0.8 | 97.01 | 0.00 |
| 2.1 | 103.2 | 6.4 | .0.9 | 97.01 | 0.00 | 0.9 | 97.01 | 0.00 |
| 2.3 | 109.3 | 12.6 | 1.0 | 97.02 | 0.01 | 1.0 | 97.01 | 0.00 |

**Table 3.** Minimum total annual cost of U$_1$ (top entry) and U$_2$

| $\pi \backslash \lambda$ | 8 | 10 | 20 | 30 | 40 | 100 |
|---|---|---|---|---|---|---|
| | 29.7 | 33.7 | 50.4 | 64.0 | 76.1 | 132.2 |
| 1 | 29.7 | 33.7 | 50.4 | 64.0 | 76.1 | 132.1 |
| | 45.3 | 50.9 | 73.1 | 89.8 | 104.3 | 166.7 |
| 9 | 44.5 | 50.1 | 71.9 | 88.4 | 102.9 | 162.8 |
| | 54.3 | 64.0 | 89.5 | 108.7 | 124.0 | 197.2 |
| 99 | 58.8 | 60.5 | 83.4 | 100.7 | 115.3 | 182.0 |

**Table 4.** Minimum total annual cost for U$_1$ (top entry) and U$_2$

| $\pi \backslash \lambda$ | 8 | 10 | 20 | 40 | 100 |
|---|---|---|---|---|---|
| | 29.7 | 33.7 | 50.4 | 76.1 | 132.2 |
| 1.5 | 29.7 | 33.7 | 50.4 | 76.1 | 132.1 |
| | 40.5 | 44.9 | 62.9 | 89.0 | 141.9 |
| 10 | 39.5 | 44.2 | 62.2 | 88.4 | 141.6 |
| | 50.9 | 55.3 | 75.8 | 103.3 | 161.4 |
| 100 | 48.9 | 53.2 | 73.5 | 101.1 | 157.5 |

**Table 5.** Minimum total annual cost of U$_2$ (top entry) and U$_3$

| $\pi \backslash \lambda$ | 8 | 10 | 30 | 100 |
|---|---|---|---|---|
| | 44.5 | 50.1 | 88.4 | 162.8 |
| 9 | 44.4 | 50.1 | 88.4 | 161.7 |
| | 54.8 | 60.5 | 100.7 | 182.0 |
| 99 | 54.7 | 60.5 | 100.7 | 181.8 |

## 5. Comparison of the Review Methods

Considering the three review methods, the minimum total annual cost is used for comparing their performance. Table 3 compares U1 with U2 for different values of λ and π. In this comparison ν and ω have been set to their optimal values. We see that there is no practical difference for small π (relative to h) and low value of λ. For large π and λ, U$_2$ is a more desirable method.

Table 4 also compares U$_1$ with U$_2$ for different values of π and λ. Here we observe that the difference is not much, even for large values of ˆπ and λ.

Table 5 demonstrates that the extra complication of U$_3$ does not yield considerable improvement with respect to U$_2$.

## 6. Conclusion

Generally, in a consideration of different review times, the periodic review method should be compared with U$_2$. There is no great practical and computational difficulty imposed by using the first review period different from the subsequent ones, but there is a possibility of a significant reduction in the total annual cost.

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