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On Commutative Reduced Baer Rings

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Abstract

It is shown that a commutative reduced ring R is a Baer ring if and only if it is a CS-ring; if and only if every dense subset of Spec (*R*) containing Max (*R*) is an extremally disconnected space; if and only if every non-zero ideal of R is essential in a principal ideal generated by an idempotent.

Keywords: Extremmally disconnected; Baer ring; Zariski topology

1. Introduction

Al-Ezeh [1], Azarpanah and Karamzadeh [4] have given some algebraic characterizations for externally disconnected spaces. In particular, they have proved independently that C(X) is a Baer ring if and only if X is and extremally disconnected space. In this paper we generalize this theorem for reduced rings and we give several equivalent conditions for reduced Baer rings.
Throughout, *R* is a commutative ring with identity. We say that *R* is a reduced ring if *R* has no non-zero nilpotent elements. Also *R* is called a Baer ring if the annihilator of each ideal I in R, Ann(I), is generated by an idempotent. If I and J are ideals in *R*, we say *I* is essential in *J* if $I \subseteq J$ and every non-zero ideal inside *J* intersects *I* non-trivially, and when we say *I* is essential we mean it is essential in *R*. An ideal *I* in *R* is called a closed ideal if it is not essential in a larger ideal, and a ring *R* is said to be a CS-ring if every closed ideal is a direct summand [7]. It is trivial to see that if *I* is an ideal in a reduced ring *R*, then $I \oplus Ann_R(I)$ is an essential ideal in *R* and therefore *I* is an essential ideal if and only if $\mathrm{Ann}_R(I)=(0)$.

We denote Spec (*R*) and Max (*R*) for the spaces of prime ideals and maximal ideals, respectively. For any $a \in R$ and any ideal *I* of *R*, we set

$$V(a) = \{P \in Spec(R) : a \in P\}$$

and

$$V(I) = \bigcap_{a\in I} V(a) = \{P \in Spec(R) : I \subseteq P\}.$$

Then $V(I) \cup V(J) = V(I \cap J) = V(IJ)$, for all ideals I and J of R. Also for any family $\{I_k\}_{k\in K}$ of ideals we have: $\bigcap_{k\in K} V(I_k) = V(\sum_{k\in K} I_k)$. From this it follows that $\mathcal{F}$={V(*I*): *I* is an ideal of *R*} is closed under finite union and arbitrary intersections, so that there is a topology on spec (*R*) for which $\mathcal{F}$ is the family of closed sets. This is called the Zariski topology [6]. If $S \subseteq$ Spec (*R*), we put $V_s(a) = V(a) \cap S$, $V_S(I) = V(I) \cap S$. We consider S as a subspace of Spec (*R*).

Throughout, X will denote a completely regular Hausdorff space and C(X) denotes the ring of continuous real-valued functions on X. A space X is said to be extremally disconnected if every closed set has a closed interior or equivalently, every open set has an open closure [5].

2. Baer Rings

Throughout this section S is a dense subspace of

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Spec (R), i.e., $\cap S = (0)$. The operators cl and int denote the closure and the interior in S. We first need the following lemmas:

Lemma 2.1. Let R be a reduced ring, S be a dense subset of Spec (R) and a, b $\in$ R. Then $\operatorname{int} V_S(a) \subseteq$ int $V_S(b)$ if and only if Ann(a) $\subseteq$ Ann(b).

Proof. Let $\operatorname{int} V_S(a) \subseteq$ int $V_S(b)$ and c $\in$ Ann(a), then ac=0 implies that

$$S - V_S(c) \subseteq \operatorname{int} V_S(a) \subseteq \operatorname{int} V_S(b) \subseteq V_S(b)\ .$$

This means that bc=0 and therefore c $\in$ Ann(b). Conversely, let Ann(a) $\subseteq$ Ann(b). Let P $\in$ int V_S(a) and P $\notin$ V_S(b) and get a contradiction. Now $P \notin S -$ int V_S(a) implies that there is $0 \neq c \in R$ with S- int V_S(a) $\subseteq$ V_S(c) and $c \notin P$. Clearly ac=0 and bc $\neq$ 0. Then $c \in$ Ann(a) and $c \notin$ Ann(b) which is a contradiction. □

We know that a subset A of the space X is clopen (closed and open) if and only if there exists $f \in C(X)$ such that $f = 0$ on A and $f = 1$ on X-A [5]. We also need the following lemma.

Lemma 2.2. Let R be a reduced ring and Max $(R) \subseteq S$. Then A is a clopen subset of S if and only if there exists an idempotent $e \in R$ such that A=V_S(e).

Proof. Suppose that A is a clopen subset of S, $I = \cap A$ and $J = \cap A^c$, then A=clA=$V_S(\cap A)$ =Vs(I) and A^c=V_S(J) and $V_S(I) \cap V_S(J) = \phi$. Hence I+J=R, so there exist $e \in I$ and $e' \in J$ such that $e + e' = 1$. On the other hand, $V_S(e) \cup V_S(e') = S$ implies that $ee' = 0$, $i.e., e^2 = e$. Consequently, A=V_S(I)=V_S(e). The converse is trivial.

The structure of essential ideals of C(X), have been studied before [2,3] and a topological characterization of essential ideals of C(X) was given. In the following lemma we characterize the essential ideals of reduced ring R via a topological property.

Lemma 2.3. let R be a reduced ring, I be a non-zero ideal of R and let S be a dense subset of Spec (R). Then I is an essential ideal in R if and only if int V_S(I)= ϕ.

Proof. Suppose the interior of V_S(I) is not empty and denoted by U=intV_S(I). Let $P \in U$. Since S-U is closed, there exist $a \in \bigcap_{P' \in S-U} P' - P$. Thus for every $b \in I$, ab=0, i.e., Ann(I) $\neq$ (0), a contradiction.

Conversely, let K be a non-zero ideal in R and $0 \neq b \in K$, then S-V_S(b) is open set and clearly $(S - V_S(b)) \cap (S - V_S(I)) \neq \phi$, so there is $a \in I$ such that $(S - V_S(b)) \cap (S - V_S(a)) \neq \phi$, hence $V_S(ab) \neq S$, *i.e.*, $0 \neq ab \in K \cap I$. □

Now we give the main result of this paper.

Theorem 2.4. Let R be a reduced ring and let Max $(R) \subseteq S$ be dense subset of Spec(R). The following statements are equivalent.

(1) S is extremally disconnected.

(2) R is a Baer ring.

(3) Every non-zero ideal in R is essential in a principle ideal generated by an idempotent.

(4) R is a CS-ring.

Proof. (1)$\Rightarrow$ (2) Let T be any subset of R, we are to show that Ann(T)=(e), where e=e^2. put F= int $\bigcap_{a \in T} V_S(a)$. According to (1), F is a clopen subset of S. If F= ϕ, we put I=(T) and we have $V_S(I) = \bigcap_{a \in I} V_S(a) = \bigcap_{a \in T} V_S(a)$. Hence $F = \operatorname{int} \cap V_S(a) = \phi$, which means that I is an essential ideal in R, by Lemma 2.3. Thus Ann(T)=Ann(I)=(0) and we are through. Hence we may assume that F= ϕ. According to Lemma 2.2, there exists an idempotent $e \in R$ with F=V_S(e) and S-F=V_S(1-e). We claim that Ann(T)=(1-e). To see this, let $b \in$ Ann(t), then $ab = 0$, $\forall a \in T$ implies that $S - V_S(b) \subseteq V_S(a)$, $\forall a \in T$. Thus $S - V_S(b) \subseteq \operatorname{int} \bigcap_{a \in T} V_S(a) = F = V_S(e)$. This means that $S = V_S(b) \cup V_S(e) = V_S(be), i.e., be = 0$ and therefore $b \in (1 - e)$. Conversely, we note that

$$\operatorname{int} V_S(a) \supseteq F = V_S(e) = \operatorname{int} V_S(e), \forall a \in T$$

and therefore by Lemma 2.1., $Ann(a) \supseteq Ann(e)$ $= (1 - e)$, $\forall a \in T$. This shows that $Ann(T) \supseteq (1 - e)$ and we are through.

(2)$\Rightarrow$ (3) Let I be an ideal of R, then by (2), we have Ann(I) (e) =Ann(1-e), where e=e^2. So I is essential in (1-e).

(3)$\Rightarrow$ (4) Let I be a closed ideal, then by (3), I is essential in (e), for some idempotent $e \in R$. But since I is closed we must have I=(e).

(4)$\Rightarrow$ (1) We note that (4) immediately implies (2), for if T is a subset of R, then the ideal I=Ann (T) is a closed ideal in R. To see this, we let I be essential in a larger ideal J, then $TJ \neq (0)$ implies that s. But R is reduced ring and hence $TJ \bigcap I = (0)$, which is impossible. This shows that I=Ann(T) is a closed ideal and by (4), I is generated by an idempotent. Now we assume (2) and show that for any closed set F, the interior of F is closed (note, we assume that int $F \neq \phi$). Since F is closed, then $F \bigcap_{a\in T} V_S(a)$, where T is some index set. Hence by (2), we have

$$Ann(T) = \bigcap_{a\in T} Ann(a) = (1-e),$$

where e=e^2. We claim that int F=V_S(e), to see this let $P \in \text{int F}$, then there exists $b \in R$ with $S - \text{int } F \subseteq V_S(b)$ and $b \notin P$. Now we have

$$P \in S - V_S(b) \subseteq \text{int } F \subseteq \bigcap_{a\in T} V_S(a).$$

Hence $S - V_S(b) \subseteq V_S(a)$, $\forall a \in T$. Therefore ab=0, $\forall a \in T$, which means that $b \in Ann(T) = (1-e)$. Thus $P \notin V_S(b)$ implies that $P \notin V_S(e-1)$ and therefore $P \in V_S(e), i.e., \text{int } F \subseteq V_S(e)$. Now suppose that $P \in V_S(e)$, there exists $b \in R$ such that $S - V_S(e) \subseteq V_S(b)$ and $P \notin V_S(b)$.

Then be=0 implies that $b \in (1-e) = Ann(T)$ $= \bigcap_{a\in T} Ann(a)$. Thus ab=0, $\forall a \in T$ and therefore $S - V_S(b) \subseteq \bigcap_{a\in T} V_S(a)$ which means that $P \in$ $S - V_S(b) \subseteq, \text{int} \bigcap_{a\in T} V_S(a), i.e., V_S(e) \subseteq \text{int F}$. This proves our claim and we are through. □

The following result is well-known, see Theorem 3.6. in [4].

Corollary 2.5. The following statements are equivalent.

(1) X is extremally disconnected.

(2) C (X) is a Baer ring.

(3) Every non-zero ideal in C(X) is seeential in a principle ideal generated by an idempotent.

(4) C(X) is a CS-ring.

Proof. It is well-Known that Max $(C(X)) \cong \beta X$, where βX is the stone-Cech compactification of X [5]. We note that X is extremally disconnected if and only if βX is extremally disconnected. Hence the corollary follows from Theorem 2.4, by letting S=Max (C(X)). □

For a ring R, let B(R) be the set of idempotents of R. It is well-known that B(R) can be made a Boolean algebra. Also it should be recalled that B(R) is complete if every subset has either infimum or supremum.

Proposition 2.6. Let R be a reduced ring and Max $(R) \subseteq S$. Then B(R) is complete if and only if the union of any collection of clopen subsets of S is clopen.

Proof. Suppose the union of any collection of clopen subsets if S is clopen. Let $B = \{e_k : k \in K\}$ be any subset of B(R). By Lemma 2.2., $V_S(e_k)$ is clopen, $\forall a \in K$. Hence $A = \bigcup_{k\in K} V_S(e_k)$ is clopen, so there exists $e \in R$ such that $A = V_S(e)$. Obviously e is the infimum of B. Conversely, let $\{A_k : k \in K\}$ be any collection of clopen sets. Then by Lemma 2.2., there exist the idempotent elements $e_k \in R$ such that $A_k = V_S(e_k)$. Let $e = in\, f\{e_k : k \in K\}$. We have $V_S(e) = \bigcup_{k\in K} A_k$. Therefore $\bigcup_{k\in K} A_k$ is clopen. □

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