Journal of Sciences, Islamic Republic of Iran 15(2): 163-169 (2004)
University of Tehran, ISSN 1016-1104

On Two-parameter Dynamical Systems and Applications

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Abstract

In this note some useful properties of strongly continuous two-parameter semigroups of operators are studied, an exponential formula for two-parameter semigroups of operators on Banach spaces is obtained and some applied examples of two-parameter dynamical systems are discussed.

Keywords: Two-parameter semigroup; Generator

1. Introduction

Let X be a Banach space and $B(X)$ be the set of all bounded linear operators on X. Recall that a homomorphism $t \to u(t)$ from $(R,+)$ into $B(X)$ is called one-parameter semigroup if $u(0) = I$. A semigroup $\{u(t)\}_{t\geq 0}$ is called strongly continuous (or C_0-semigroup) if $t \to u(t)x$ is continuous for every x in X and is called uniformly continuous if $u \to u(t)$ is norm continuous. The infinitesimal generator H of $\{u(t)\}_{t\geq 0}$ is defined by:

$$H(x) =: \lim_{t\to 0}\frac{u(t)x - x}{t}, \quad x \in D(H) \tag{1}$$

where $D(H) = \left\{ x \in X \ : \lim_{t\to 0}\frac{u(t)x - x}{t} \text{ exists} \right\}$.

By a two-parameter dynamical system on X we mean a function $(s,t) \mapsto W(s,t)x$ from $R_+ \times R_+$ into $B(X)$ such that:

i) $W(s+s', t+t') = W(s,t)W(s',t')$

ii) $W(0,0) = I$.

It is called strongly continuous if $(s,t) \to W(s,t)x$ is continuous for all x in X and is called uniformly continuous if $(s,t) \to W(s,t)$ is norm continuous.

To any two-parameter dynamical system $W(s,t)$ we associate two one-parameter semigroups $u(s) = W(s,0)$ and $v(t) = W(0,t)$, the semigroup property of W implies that $W(s,t) = u(s)v(t)$. One can see that $W(s,t)$ is strongly (resp. uniformly) continuous if and only if $u(s)$ and $v(t)$ are strongly (resp. uniformly) continuous.

The infinitesimal generator of $u(s)$ and $v(t)$ are denoted by H_1 and H_2, respectively. We will think of the pair (H_1, H_2) as the infinitesimal generator of $W(s,t)$.

The theory of n-parameters semigroups of operators which is an extension of one-parameter case developed by E. Hille in 1944. In 1946 N. Dunford and I.S. Segal [3] applied the concept to prove the theorem of Weierstrass. In [5] Hille and Phillips studied X-parameter semi-groups of operators which is an extension of n-parameter semi-groups. O.A. Ivanova [6]

2000 AMS subject classification: 46N50, 47D05, 46D06.

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obtained some other results in n-parameters semi-groups in 1966. In section (2) we state and prove some useful lemmas which provide some necessary and sufficient conditions for the product of two one-parameter dynamical systems to be a two-parameter one, as a consequence of these lemmas we extend the Hille-Yosida theorem for two-parameter case. Also an exponential formula for strongly continuous dynamical systems is proved. In section (3) we mention some classes of two-parameter dynamical systems which can be used for some counter examples. These examples provide some applications of this subject.

In [7] we have applied the theory of n-parameters semigroups of operators for generalizing the abstract Cauchy problem to n-parameter case. One can see that the semigroups of operators arise naturally in several areas of applied mathematics including prediction theory of random fields [8,12]. Such a semigroup of operators can be used to describe the time evolution of a physical system in quantum field theory, statistical mechanics and partial differential equations [2,9-11].

2. Main Results

The Hille-Yosida theorem in one-parameter case has a principal role in semigroup theory. We need the following lemmas to prove similar Hille-Yosida theorem for two-parameter case. In these lemmas we study the elementary properties of two-parameter dynamical systems and their relations with one-parameter semigroups.

Lemma 2.1. (see [10].I.5.5). Let A be the infinitesimal generator of a C_0-semigroup $T(t)$. If A_λ is the Yosida-approximation of A, *i.e.* $A_\lambda = \lambda A R(\lambda, A)$ then $T(t)x = \lim\limits_{\lambda\to\infty} \exp(tA_\lambda)$, where $R(\lambda, A) = (\lambda I - A)^{-1}$.

The following lemma proves a useful property of two-parameter dynamical systems.

Lemma 2.2. Let $W(s,t)$ be a C_0-two-parameter semigroups on Banach space X with the infinitesimal generators (H_1, H_2), then $D(H_i)\bigcap D(H_iH_j) \subseteq D(H_jH_i)$, $(i, j = 1,2)$, and for, $x \in D(H_i)\bigcap D(H_iH_j)$

$$H_iH_j(x) = H_jH_i(x)$$

Proof. Let $u(s) = W(s,0)$ and $v(t) = W(0,t)$. The semigroup property of $W(s,t)$ implies that

$$\begin{aligned} u(s)v(t) &= W(s,t) = W(0+s, t+0) \\ &= W(0,t)W(s,0) = v(t)u(s) \end{aligned} \tag{2}$$

hence $u(s)v(t) = v(t)u(s)$ for all $s,t \geq 0$. Now let x be in $D(H_1)\bigcap D(H_1H_2)$ then

$$\begin{aligned} H_1H_2(x) &= \lim_{s\to 0}\frac{u(s)H_2(x) - H_2(x)}{s} \\ &= \lim_{s\to 0}\frac{1}{s}\left(u(s)(\lim_{t\to 0}\frac{v(t)x - x}{t}) - \lim_{t\to 0}\frac{v(t)x - x}{t}\right) \\ &= \lim_{s\to 0}\lim_{t\to 0}\frac{1}{st}(u(s)v(t)x - u(s)x - v(t)x + x) \\ &= \lim_{s\to 0}\lim_{t\to 0}\frac{1}{st}(v(t)u(s)x - v(t)x - u(s)x + x), \\ &\qquad \text{since} \quad u(s)v(t) = v(t)u(s) \\ &= \lim_{s\to 0}\lim_{t\to 0}\frac{1}{t}\left(v(t)(\frac{u(s)x - x}{s}) - \frac{u(s)x - x}{s}\right) \\ &= \lim_{s\to 0} H_2(\frac{u(s)x - x}{s}) = H_2H_1(x). \end{aligned}$$

The last equality holds since $x \in D(H_1)$ and H_2 is a closed operator by the Hille-Yosida theorem. Thus $H_1H_2 = H_2H_1$. □

For C_0-two-parameter semigroup $W(s,t) = u(s)v(t)$, we know that ([10].I.2.2) there exist $\omega, \omega' > 0$ and $M_1, M_2 \geq 1$ s.t. $\|u(s)\| \leq M_1 e^{\omega s}$ and $\|v(t)\| \leq M_2 e^{\omega' t}$. Hence if $M = M_1M_2$, trivially $\|W(s,t)\| \leq M\, e^{\omega s + \omega' t}$.

The next lemma provides a sufficient and necessary condition for the product of two one-parameter semigroups to be a two-parameter semigroup. Using this lemma we can extend many of well-known results to two-parameter semigroups.

Lemma 2.3. (a) Suppose $\{u(s)\}_{s\geq 0}$ and $\{v(t)\}_{t\geq 0}$ are two C_0-one-parameter semigroups of operators on Banach space X with the infinitesimal generator H_1 and H_2 respectively, then $W(s,t) = u(s)v(t)$ is a C_0-two-parameter semigroup of operators if and only if there is an $\omega > 0$ such that for each $i = 1,2$, $[0,\infty) \subseteq \rho(H_i)$ and for each $\lambda,\ \lambda' \geq \omega$, we have

$$R(\lambda', H_1)R(\lambda, H_2) = R(\lambda, H_2)R(\lambda', H_1).$$

(b) If the one-parameter semi-groups $\{u(s)\}_{s\geq 0}$ with the generator H_1 is strongly continuous and $\{v(t)\}_{t\geq 0}$ with the generator H_2 is uniformly continuous, then $W(s,t)=u(s)v(t)$ is a C_0-two-parameter semigroup of operators if and only if $H_1H_2=H_2H_1$.

Proof. (a) First suppose W is a C_0-two-parameter semigroup of operators. Since H_1 and H_2 are the infinitesimal generator of $\{u(s)\}_{s\geq 0}$ and $\{v(t)\}_{t\geq 0}$, respectively, by the Hille-Yosida Theorem ([10].I.5.3), there is an $\omega_1,\omega_2>0$ such that for each $\lambda\geq\omega$ and $\lambda'>\omega$, $R(\lambda,H_1)$ and $R(\lambda',H_2)$ exist and are bounded operators. Let $\omega=\max\{\omega_1,\omega_2\}$. If $\lambda\geq\omega$ from [10].I.5.4

$$R(\lambda,H_1)(x)=\int_0^\infty e^{-\lambda t}u(s)xds \ \ \&$$

$$R(\lambda',H_2)(x)=\int_0^\infty e^{-\lambda' t}v(t)xds$$

Also we know

$$\begin{aligned}u(s)v(t)&=W(s,0)W(0,t)\\&=W(0,t)W(s,0)=v(t)u(s)\end{aligned}$$

so

$$\begin{aligned}R(\lambda,H_1)(v(t)x)&=\int_0^\infty e^{-\lambda s}u(s)v(t)xds\\&=\int_0^\infty e^{-\lambda s}v(t)u(s)xds\\&=v(t)\int_0^\infty e^{-\lambda s}u(s)xds\\&=v(t)R(\lambda,H_1)x.\end{aligned}$$

Now let $\lambda\geq\omega$, we know $R(\lambda,H_i)$, $i=1,2$, is bounded so

$$\begin{aligned}R(\lambda,H_1)R(\lambda',H_2)x&=R(\lambda,H_1)\int_0^\infty e^{-\lambda' t}v(t)xdt\\&=\int_0^\infty e^{-\lambda' t}v(t)R(\lambda,H_1)xdt\\&=R(\lambda',H_2)R(\lambda,H_1)x.\end{aligned}$$

and this proves the necessary part of lemma.

For the converse suppose there is an $\omega>0$ such that for each $\lambda,\lambda'\geq\omega$, $R(\lambda,H_1)$ and $R(\lambda',H_2)$ exist and commute. So we have $H_\lambda^1H_{\lambda'}^2=H_{\lambda'}^2H_\lambda^1$ where $H_\lambda^1=\lambda^2R(\lambda,H_1)-\lambda I$ and $H_{\lambda'}^2=\lambda^2R(\lambda',H_2)-\lambda I$ are the Yosida approximations of H_1 and H_2, respectively. Applying Lemma 1.1 we have $u(s)x=\lim_{\lambda\to\infty}e^{sH_\lambda^1}x$ and $v(t)x=\lim_{\lambda'\to\infty}e^{tH_{\lambda'}^2}x$, thus

$$\begin{aligned}u(s)v(t)x&=\lim_{\lambda\to\infty}e^{sH_\lambda^1}v(t)x\\&=\lim_{\lambda\to\infty}\lim_{\lambda'\to\infty}e^{sH_\lambda^1}e^{tH_{\lambda'}^2} && (e^{sH_\lambda^1}\text{ is continuous})\\&=\lim_{\lambda\to\infty}\lim_{\lambda'\to\infty}e^{tH_{\lambda'}^2}e^{sH_\lambda^1}x && (\text{since } H_\lambda^1H_{\lambda'}^2=H_{\lambda'}^2H_\lambda^1)\\&=\lim_{\lambda\to\infty}v(t)e^{sH_\lambda^1}x\\&=u(s)v(t) && (v(t)\text{ is continuous})\end{aligned}$$

Hence $W(s,t)=u(s)v(t)$ is a C_0-two-parameter semigroup of operators.

(b) If W is a C_0-two-parameter semigroup of operators, then by the previous lemma, the equality $H_1H_2=H_2H_1$ holds. Conversely, we know H_2 is bounded and $H_1H_2=H_2H_1$ (note that the equality $H_1H_2=H_2H_1$ defines only on $D(H_1)$). Let $\|u(s)\|\leq Me^{\omega s}$, for some $M\geq 1$ and $\omega>0$. Hence by the Hille-Yosida theorem, for $\lambda\geq\omega$, $R(\lambda,H_1)$ exists. If $\lambda,\lambda'\geq\omega$, then

$$(\lambda I-H_1)(\lambda' I-H_2)=(\lambda' I-H_2)(\lambda I-H_1),$$

since $H_1H_2=H_2H_1$. Also $(\lambda I-H_1)D(H_1)=X$, thus

$$(\lambda' I-H_2)(\lambda I-H_1)D(H_1)=(\lambda' I-H_2)X=X.$$

Now let $y\in X$, so $y=(\lambda' I-H_2)(\lambda I-H_1)x$, for some $x\in D(H_1)$. But from our hypothesis,

$$\begin{aligned}y&=(\lambda' I-H_2)(\lambda I-H_1)x\\&=(\lambda I-H_1)(\lambda' I-H_2)x\end{aligned}$$

hence

$$R(\lambda',H_2)R(\lambda,H_1)y=x=R(\lambda,H_1)R(\lambda',H_2)y.$$

This proves the equality.

The previous part of this theorem completes the proof of part (b). □

We are ready to state an extension of Hille-Yosida theorem as follows:

Theorem 2.4. A pair (H_1,H_2) of operators with domains in X is the infinitesimal generator of a C_0-two-parameter semigroup $W(s,t)$ satisfying $\|W(s,t)\| \le M_0 e^{\omega s+\omega' t}$, for some $M_0 \ge 1$, $\omega, \omega' > 0$ if and only if

(i) H_1 and H_2 are closed and densely defined operators and

$$R(\lambda, H_2)R(\lambda, H_1) = R(\lambda, H_1)R(\lambda, H_2).$$

for each $\lambda \ge \omega, \lambda' \ge \omega'$.

(ii) The resolvent sets $\rho(H_1)$ and $\rho(H_2)$ contain $[\omega, \infty)$ and $[\omega', \infty)$, respectively and there is some $M \ge 1$ such that

$$\left\|R(\lambda, H_1)^n\right\| \le \frac{M}{(\operatorname{Re}\lambda - \omega)^n},$$

$$\left\|R(\lambda', H_2)^n\right\| \le \frac{M}{(\operatorname{Re}\lambda' - \omega')^n}$$

where $\operatorname{Re}\lambda \ge \omega$ and $\operatorname{Re}\lambda' \ge \omega'$.

Proof. Let (H_1, H_2) be the infinitesimal generator of $W(s,t)$ satisfying $\|W(s,t)\| \le M_0 e^{\omega s+\omega' t}$, so H_1 is the infinitesimal generator of $u(s)$ satisfying $\|u(s)\| = \|W(s,0)\| \le M_0 e^{\omega s}$ and H_2 is the infinitesimal generator of $v(t)$ satisfying $\|v(t)\| = \|W(0,t)\| \le M_0 e^{\omega' t}$. By Lemma 2.3,

$$R(\lambda, H_2)R(\lambda, H_1) = R(\lambda, H_1)R(\lambda, H_2).$$

Using the Hille-Yosida theorem ([10].I.3.1), we conclude that **(i)** and **(ii)** are valid.

Conversely, from conditions **(i), (ii)** and by the Hille-Yosida theorem there exist C_0-one-parameter semigroups $u(s)$ and $v(t)$ satisfying $\|u(s)\| \le Me^{\omega s}$ and $\|v(t)\| \le Me^{\omega' t}$, with the infinitesimal generators H_1 and H_2, respectively. Now define $W(s,t) = u(s)v(t)$, then by Lemma 2.3 $W(s,t)$ is a C_0-two-parameter semigroup and $\|W(s,t)\| \le M_0 e^{\omega s+\omega' t}$, where $M_0 = M^2$, and this completes the proof. □

The following theorem establishes an exponential formula for strongly continuous dynamical systems.

Theorem 2.5. Let $W(s,t)$ be a C_0-two-parameter semigroup with the infinitesimal generator (H_1, H_2) then

$$W(s,t)x = \lim_{n\to\infty}(I - \frac{s}{n}H_1)^{-n}(I - \frac{t}{n}H_2)^{-n}(x)$$

and the limit is uniform in (s,t) on any compact subset of R_+^2.

Proof. Without loss of generality we prove theorem for the compact set $[0,S]\times[0,T]$. From ([10].I.2.2) there exist $M_1, M_2 \ge 1$ and $\omega_1, \omega_2 > 0$ such that $\|u(s)\| \le M_1 e^{\omega_1 s}$ and $\|v(t)\| \le M_1 e^{\omega_2 t}$, thus for each $(s,t) \in [0,S]\times[0,T]$, $\|u(s)\| \le M_1 e^{\omega_1 S}$ and $\|v(t)\| \le M_1 e^{\omega_2 T}$. By ([10].I.8.3), we know that

$$u(s)x = \lim_{n\to\infty}(I - \frac{s}{n}H_1)^{-n}(x),$$
$$v(t)x = \lim_{n\to\infty}(I - \frac{t}{n}H_2)^{-n}(x) \tag{3}$$

and the limits are uniform in s and t on $[0,S]$ and $[0,S]$, respectively. Let $\varepsilon > 0$ be given and $\varepsilon' = \frac{\varepsilon}{3M_1 e^{S\omega_1}}$, then there is a natural number N_1 such that

$$\| \ v(t)x - (I - \frac{t}{m}H_2)^{-m}(x) \ \| < \varepsilon' \tag{4}$$

for all $m \ge N_1$ and $t \in [0,T]$. On the other hand, by the Proposition 2.3 we have

$$\| \ (I - \frac{s}{n}H_1)^{-n} \ \| = \| \ (\frac{n}{s}R(\frac{s}{n}, H_1))^n \ \|$$

$$\le (\frac{n}{s})^n \frac{M_1}{(\frac{n}{s} - \omega_1)^n} = (\frac{\frac{n}{s}}{\frac{n}{s} - \omega_1})^n M_1$$

$$\le (\frac{\frac{n}{S}}{\frac{n}{S} - \omega_1})^n M_1,$$

for sufficient large n and all $0 < s < S$. But

$$\lim_{n\to\infty}(\frac{\frac{n}{S}}{\frac{n}{S} - \omega_1})^n = \lim_{n\to\infty}(1 + \frac{S\omega_1}{n - S\omega_1})^n$$

$$= \lim_{n\to\infty}(1 + \frac{S\omega_1}{n})^{n+S\omega_1} = e^{S\omega_1}.$$

Hence $\left(\dfrac{\frac{n}{S}}{\frac{n}{S}-\omega_1}\right)^n \le M'$, for some $M'>0$, and so

$$\| (I-\frac{s}{n}H_1)^{-n} \| \le \left(\dfrac{\frac{n}{S}}{\frac{n}{S}-\omega_1}\right)^n M_1 \le M_3,$$

for all $s\in[0,S]$,

where $M_3 = M_1M'$. Similarly there exists $M_4>0$ such that for all $t\in[0,T]$,

$$\| (I-\frac{s}{n}H_1)^{-n} \| \le M_4 .$$

Now from (3), for $\varepsilon'' = \dfrac{\varepsilon}{3M_3}$

$\exists N_2>0$ s.t. $\forall m,n \ge N_2$,

$$\| (I-\frac{t}{m}H_2)^{-m}(x) - (I-\frac{t}{n}H_2)^{-n}(x) \| < \varepsilon'',$$

$$t\in[0,T] \tag{5}$$

and for $\varepsilon''' = \dfrac{\varepsilon}{3M_4}$

$\exists N_3>0$ s.t. $\forall n\ge N_3$,

$$\| u(s)x - (I-\frac{s}{n}H_1)^{-n}(x) \| < \varepsilon''',$$

$$s\in[0,S] \tag{6}$$

Let $N=\max\{N_1,N_2,N_3\}$, $(s,t)\in[0,S]\times[0,T]$ and $n\ge N$, by (4), (5), (6) and the equality $u(s)(I-\frac{t}{N_2}H_2)^{-N_2} = (I-\frac{t}{N_2}H_2)^{-N_2}u(s)$, we have

$$\left\|W(s,t) - (I-\frac{s}{n}H_1)^{-n}(x)(I-\frac{t}{n}H_2)^{-n}(x)\right\|$$

$$\le \left\|u(s)v(t)x - u(s)(I-\frac{t}{N_2}H_2)^{-N_2}\right\|$$

$$+\left\|u(s)(I-\frac{t}{N_2}H_2)^{-N_2}(x) - (I-\frac{s}{n}H_1)^{-n}(I-\frac{t}{N_2}H_2)^{-N_2}(x)\right\|$$

$$+\left\|(I-\frac{s}{n}H_1)^{-n}(I-\frac{t}{N_2}H_2)^{-N_2}(x) - (I-\frac{s}{n}H_1)^{-n}(I-\frac{t}{n}H_2)^{-n}(x)\right\|$$

$$\le \|u(s)\|\left\|v(t) - (I-\frac{t}{N_2}H_2)^{-N_2}(x)(I-\frac{t}{n}H_2)^{-n}(x)\right\|$$

$$+\left\|(I-\frac{t}{N_2}H_2)^{-N_2}\right\|\left\|u(s)x - (I-\frac{s}{n}H_1)^{-n}(x)\right\|$$

$$+\left\|(I-\frac{s}{n}H_1)^{-n}\right\|\left\|(I-\frac{t}{N_2}H_2)^{-N_2}(x) - (I-\frac{t}{n}H_2)^{-n}(x)\right\|$$

$$< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon.$$

This completes the proof of theorem. □

3. Applicable Examples

Let Ω be a locally compact Hausdorff space and $C_0(\Omega)$ be the set of all continuous complex-valued functions on Ω vanishing at infinity. Let q_1 and q_2 be two continuous functions on Ω such that $\sup_{x\in\Omega}\operatorname{Re} q_i(x) < \infty$ for $i=1,2$. If for s and t in R_+, $e^{sq_1+tq_2}(x) = e^{sq_1(x)+tq_2(x)}$ and $T_{q_1,q_2}(s,t): C_0(\Omega)\to C_0(\Omega): f \mapsto e^{sq_1(x)+tq_2(x)}f$, then it is not hard to see that $T_{q_1,q_2}(s,t)$ is well-defined and $\{T_{q_1,q_2}(s,t)\}_{(s,t)\in R_+^2}$ is a two-parameter semigroup which is called multiplication semigroup on $C_0(\Omega)$. The following proposition show that with changing of the q_i, $i=1,2$, we can construct many different two-parameter semigroups.

Proposition 3.1. With the above assumptions,

(a) $\{T_{q_1,q_2}(s,t)\}_{(s,t)\in R_+^2}$ is a strongly continuous two-parameter semigroup.

(b) The generator of $\{T_{q_1,q_2}(s,t)\}_{(s,t)\in R_+^2}$ is (M_{q_1}, M_{q_2}) where

$$M_{q_i}: C_0(\Omega)\to C_0(\Omega): f\mapsto q_i f .$$

(c) $\{T_{q_1,q_2}(s,t)\}_{(s,t)\in R_+^2}$ is uniformly continuous if and only if q_i 's, $i=1,2$, are bounded.

(d) $\{T_{q_1,q_2}(s,t)\}_{(s,t)\in R_+^2}$ is a contractive if and only if $\mathrm{Re}q_i(x)<0$, $i=1,2$, for all x belong to Ω.

(e) If for $s,t\geq 0$, $m_{s,t}:\Omega\to C$ is bounded functions such that $T(s,t)f=m_{s,t}f$ define a C_0-two-parameter semigroup on $C_0(\Omega)$, then there exist continuous functions $q_i:\Omega\to C$ satisfying $\sup_{x\in\Omega}\mathrm{Re}q_i(x)<\infty$, $i=1,2$, such that $m_{s,t}=e^{sq_1+tq_2}$.

Proof. **(a)** It is enough to show that the mapping $(s,t)\mapsto T_{q_1,q_2}(s,t)f$ is continuous for every f in $C_0(\Omega)$ at $(0,0)$. Let $\varepsilon>0$ be given, for $\varepsilon_0=\frac{\varepsilon}{e^{|\omega_1|+|\omega_2|+1}}\|f\|_\infty$ where $\omega_i=\sup_{x\in\Omega}\mathrm{Re}q_i(x)$, $i=1,2$ there exists a compact set $K\subseteq\Omega$ such that $|f(x)|<\varepsilon_0$, for all x in $\Omega-K$. Since and q_i, $i=1,2$ are continuous, there exist M_1 and M_2 such that $|q_i(x)|<M_i$, for all x in $\Omega-K$, let $s_0=\frac{\ln(\varepsilon+1)}{2M_1}$ and $t_0=\frac{\ln(\varepsilon+1)}{2M_2}$, then for $x\in K$ and $s<s_0$, $t<t_0$

$$\begin{aligned}|sq_1(x)+tq_2(x)|&\leq s|q_1(x)|+t|q_2(x)|\\&\leq\frac{1}{2}\ln(\varepsilon+1)+\frac{1}{2}\ln(\varepsilon+1)\\&=\ln(\varepsilon+1)\end{aligned}$$

Hence $\left|e^{sq_1(x)+tq_2(x)}-1\right|<\varepsilon$, for all $x\in K$ and $s<s_0$, $t<t_0$. And so

$$\begin{aligned}\left\|e^{sq_1+tq_2}f-f\right\|_\infty&\leq\sup_{x\in K}(\left|e^{sq_1(x)+tq_2(x)}-1\right||f(x)|)\\&\quad+(e^{|\omega_1|+|\omega_2|+1})\sup_{x\in\Omega-K}|f(x)|\\&\leq\varepsilon\|f\|_\infty+\varepsilon\|f\|_\infty\\&=2\varepsilon\|f\|_\infty\end{aligned}$$

(b) Define $T_{q_1}(s)f=e^{sq_1}f$ and $T_{q_2}(t)f=e^{tq_2}f$. Then

$$\lim_{s\to 0}\frac{T_{q_1}f(x)-f(x)}{s}=\lim_{s\to 0}\frac{e^{sq_1}f(x)-f(x)}{s}$$

$$=\lim_{s\to 0}(\frac{e^{sq_1(x)}-1}{s})f(x)=q_1(x)f(x)$$

i.e. $\frac{d}{ds}T_{q_1}(s)=M_{q_1}$, similarly $\frac{d}{dt}T_{q_2}(t)=M_{q_2}$.

(c) First we show that q_1 and q_2 are bounded if and only if M_{q_1} and M_{q_2} are bounded, for; let, q_i, $i=1,2$, be bounded then

$$\|M_{q_i}f\|_\infty=\|q_if\|_\infty\leq\|q_i\|_\infty\|f\|_\infty.$$

Hence $\|M_{q_i}\|\leq\|q_i\|_\infty$. Conversely if $\|M_{q_i}\|=m<\infty$, then for every x in Ω, by Urysohn's lemma there exists a function f_x in $C_0(\Omega)$ such that $\|f_x\|_\infty=f_x(x)=1$, this implies

$$\|M_{q_i}\|\geq\|M_{q_i}f_x\|_\infty\geq|q_i(x)f_x(x)|=|q_i(x)|$$

for all x belong to Ω.

We know that $\{T_{q_1,q_2}(s,t)\}_{(s,t)\in R_+^2}$ is uniformly continuous if and only if its generator, (M_{q_1},M_{q_2}), is bounded and hence $\{T_{q_1,q_2}(s,t)\}_{(s,t)\in R_+^2}$ is uniformly continuous if and only if q_1 and q_2 are bounded.

(d) If $\|T_{q_1,q_2}(s,t)\|\leq 1$, for all $s,t\geq 0$, then

$$\|T_{q_1,q_2}(s,0)\|\leq 1 \text{ and } \|T_{q_1,q_2}(0,t)\|\leq 1,$$

so $\|e^{sq_1}f\|_\infty\leq 1$ and $\|e^{tq_2}f\|_\infty\leq 1$ for all f in $C_0(\Omega)$. Hence $\left|e^{sq_1(x)}f(x)\right|\leq 1$ and $\left|e^{tq_2(x)}f(x)\right|\leq 1$ for all x in Ω, now let $x_0\in\Omega$, by Urysohn's lemma we can choose f in $C_0(\Omega)$ such that $\|f\|_\infty=f(x_0)=1$ thus

$$\left|e^{sq_1(x_0)}\right|\leq 1 \text{ and } \left|e^{tq_2(x_0)}\right|\leq 1$$

for all $x\in\Omega$ and $s,t\geq 0$,

this implies that $\mathrm{Re}q_i(x_0)<0$, for all x_0 in Ω.

(e) Let $m'_s=m_{s,0}$ and $m''_t=m_{0,t}$, trivially $T'(s)f=m'_sf$ and $T''(t)f=m''_tf$ define two C_0-one-parameter semigroups. It follows from [4].4.6.p.28 that there exist continuous functions $q_i:\Omega\to C$, $i=1,2$, such that

$$\sup_{x\in\Omega}\mathrm{Re}q_i(x)<\infty,\ m'_s(x)=e^{sq_1(x)}$$

and

$$m''_t(x) = e^{tq_2(x)}$$

Hence $m_{s,t}(x) = m'_s\, m''_t(x) = e^{sq_1(x)+tq_2(x)}$, and this completes the proof of proposition. □

In the special case if $\Omega = \{1, 2, ..., m\}$ trivially $C_0(\Omega) = C^m$. If $q_1 = (a_1, a_2, ..., a_m)$ and $q_2 = (b_1, b_2, ..., b_m)$ then

$$M_{q_1} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_m \end{pmatrix},$$

$$M_{q_2} = \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & b_m \end{pmatrix}$$

Hence

$$T_{q_1,q_2}(s,t) = e^{sM_{q_1}+tM_{q_2}}$$

$$= \begin{pmatrix} e^{sa_1+tb_1} & 0 & \cdots & 0 \\ 0 & e^{sa_2+tb_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & e^{sa_m+tb_m} \end{pmatrix}$$

Since q_1 and q_2 are bounded, $T_{q_1,q_2}(s,t)$ is uniformly continuous. It is easy to see that every uniformly continuous two-parameter semigroup with the infinitesimal generator (H, K) has the form e^{sH+tK}.

Remark 3.2. For two-parameter semigroup $W(s,t)$ with the infinitesimal generator (H, K), the spectral mapping theorem for one-parameter semigroup implies that

$$e^{s\sigma(H)} \subseteq \sigma(W(s,0)) \qquad e^{t\sigma(K)} \subseteq \sigma(W(0,t))$$

thus $e^{s\sigma(H)+t\sigma(K)} \subseteq \sigma(W(s,0))\sigma(W(0,t))$, but the following example shows that the inclusion

$$e^{s\sigma(H)+t\sigma(K)} \subseteq \sigma(W(s,t))$$

is not valid in general.

Let $T_{q_1,q_2}(s,t)$ be Multiplication semigroup on $C_0(\Omega)$ with the infinitesimal generator (M_{q_1}, M_{q_2}), from [4].I.4.2, we have

$$\sigma(M_{q_i}) = \overline{\{q_i(x) : x \in \Omega\}} = \overline{\{q_i(\Omega)\}}$$

and

$$s\sigma(M_{q_1}) + t\sigma(M_{q_2}) \supseteq \{sq_1(x) + tq_2(x) : x_1, x_2 \in \Omega\},$$

also

$$\sigma(T_{q_1,q_2}(s,t)) = \{\lambda : \lambda - T_{q_1,q_2}(s,t) \text{ is not invertible}\}$$
$$= \left\{e^{sq_1(x)+tq_2(x)} : x \in \Omega\right\}.$$

For appropriate $q_1 s$, q_2 and Ω, trivially $e^{s\sigma(M_{q_1})+t\sigma(M_{q_2})} \supseteq \left\{e^{sq_1(x_1)+tq_2(x_2)} : x_1, x_2 \in \Omega\right\}$ is very larger than $\sigma(T_{q_1,q_2}(s,t)) = \left\{e^{sq_1(x)+tq_2(x)} : x \in \Omega\right\}$. □

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