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On C_0-Group of Linear Operators

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Abstract

In this paper we consider C_0-group of unitary operators on a Hilbert C^*-module E. In particular we show that if $A \subseteq L(E)$ be a C^*-algebra including $K(E)$ and α_t a C_0-group of *-automorphisms on A, such that there is $x \in E$ with $<x,x>=1$ and $\alpha_t(\theta_{x,x}) = \theta_{x,x}$ $t \in R$, then there is a C_0-group u_t of unitaries in $L(E)$ such that $\alpha_t(a) = u_t\, a\, u_t^*$.

Keywords: C_0-group of linear operators; Hilbert C^*-module; Unitary operator

1 Introduction

A one parameter family $T=T(t)$, $t \in R$ of bounded operators on Banach space $X(T{:}R \to B(X))$ is called a one parameter group if it satisfies:

i) $T(0)=I$

ii) $T(s+t)=T(s)T(t)$ $t,s \in R$. Moreover if

iii) For each $x \in X$ the map $t \to T(t)x$ from R to X is continuous with respect to norm topologoy of X, then $T(t)$ is called C_0-group.

We define the infinitesimal generator of one parameter group $T(t)$ by

$$\delta x = \lim_{t \to 0} t^{-1}[T(t)x - x].$$

Where the domain $D(\delta)$ of δ is the set of all $x \in X$ such that the limit exists. See [3] Let A be a $*$-algebra. An automorphism on A is an invertible linear operator α: $A \to A$ such that $\alpha(ab)=\alpha(a)\alpha(b)$ and $\alpha(a^*) = (\alpha(a))^*$. An automorphism α on the algebra A called an inner automorphism if there is a unitary element $u \in A$ such that $\alpha(a) = uau^*$ for every $a \in A$. Suppose A is a C^*-algebra. Let E be a complex linear space which is a left A-module and $\lambda(ax)=(\lambda a)x=a(\lambda x)$ where $\lambda \in C$, $a \in A$ and $x \in E$. The space E is called a pre-Hilbert A-module if there exists an (A-valued) inner product $< , >$: $E \times E \to A$ such that for every $x, y \in E$, $\lambda \in C$ and $a \in A$, we have:

(i) $<x, x> \geq 0$

(ii) $<x, x> = 0$ iff $x = 0$

(iii) $<x + \lambda y, z> = <x, z> + \lambda <y, z>$

(iv) $<x, y> = <y, x>^*$

(v) $<ax, y> = a<x, y>$

A pre-Hilbert A-module E is called a Hilbert A-module or Hilbert C^*-module over A, if it is complete with respect to the norm $\|x\| = \|<x, x>\|^{\frac{1}{2}}$. For example if A is a C*-algebra, then A with its product as the usual action is left A-module. In addition if A equipped with the inner product $<a, b> = a\, b^*$ then it is a Hilbert A-module.

Suppose that E, F are Hilbert C^*-modules. We define $L(E,F)$ to be the set of all maps t: $E \to F$ for which there is a map t^*: $F \to E$ such that $<tx, y> = <x, t^*y>$ $(x \in E, y \in F)$. It is easy to see that t must be A-linear and bounded. We call $L(E,F)$ the set of adjointable maps

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from E to F. Thus every element of $L(E,F)$ is a bounded A-linear map. In particular, $L(E,F)$ which we abbreviate to $L(E)$ is a *-algebra. Let E, F be Hilbert C^*-modules. For x in E and y in F, define $\theta_{x,y}: F \to E$ by $\theta_{x,y}(z) = <z, y>x$ $(z \in F)$. It is easy to check that $\theta_{x,y} \in L(E,F)$ with $(\theta_{x,y})^* = \theta_{y,x}$ and also that the following relations hold: (where G is Hilbert C^*-module)

$$t\,\theta_{x,y} = \theta_{tx,y} \qquad (t \in L(E,G)),$$

$$\theta_{x,y}\, s = \theta_{x,s^*y} \qquad (s \in L(G,F)).$$

We denote by $K(F,E)$ the closed linear subspace of $L(F,E)$ spanned by $\{\theta_{x,y}\ x \in E, y \in F\}$ and we write $K(E)$ for $K(E,E)$. We also have $1x = x$ if A has an identity 1 and also $\overline{K(E)E = E}$ (See [1]). An operator $u \in L(E,F)$ is said to be unitary if $u^*u = 1_E$ and $uu^* = 1_F$. In this paper we consider Hilbert A-module E over unital C^*-algebra A. (See [1]-[3] for more result on Hilbert C^*-module and semi-group of linear operators).

2. The Main Results

Theorem 1. Let $u_t = u(t)$ be a C_0-group of unitary operators on a Hilbert C^*-module E. Then $\alpha_t(a) = u_t a u_t^* (a \in K(E))$ is a C_0-group of automorphism on $K(E)$.

Proof. It is easy to see that α_t is an automorphism and we have:

$$\begin{aligned}\alpha_t(\theta_{x,y}) - \theta_{x,y} &= u_t(\theta_{x,y})u_t^* - \theta_{x,y}\\ &= \theta_{u_t x, u_t y} - \theta_{x,u_t y} + \theta_{x,u_t y} - \theta_{x,y}\\ &= (\theta_{u_t x - x, u_t y}) + (\theta_{x, u_t y - y}).\end{aligned}$$

So we have:

$$\begin{aligned}&\|\alpha_t(\theta_{x,y}) - \theta_{x,y}\| \le \|\theta_{u_t x - x, u_t y}\| + \|\theta_{x,u_t y - y}\|\\ &\le \|u_t x - x\|\|u_t y\| + \|x\|\|u_t y - y\|\\ &= \|u_t x - x\|\|y\| + \|x\|\|u_t y - y\|.\end{aligned}$$

Since u_t is continuos then

$$\lim_{t\to 0}\|\alpha_t(\theta_{x,y}) - \theta_{x,y}\| = 0 .$$

Which shows that $\alpha_t(\theta_{x,y})$ is continuous. Since $\{\theta_{x,y}: x,y \in E\}$ is dense in $K(E)$, thus $\alpha(a)$ is continuous for each $a \in K(E)$. This shows that α_t is a C_0-group of *-automorphisms. □

Theorem 2. Let A be a C^*-algebra and $K(E) \subseteq A \subseteq L(E)$ and α_t a C_0-group of *-automorphisms on A such that there is $x \in E$ with $<x, x> = 1$ and $\alpha_t(\theta_{x,x}) = \theta_{x,x}$ $t \in R$, then there is a C_0-group u_t of unitaries in $L(E)$ such that $\alpha_t(a) = u_t a u_t^*$.

Proof. For each $a \in A$ we define $u_t(ax) = \alpha_t(a)x$. We have:

$$\begin{aligned}\|ax\| &= \|<x,x>ax\|\\ &= \|a(\theta_{x,x})x\| \le \|a\theta_{x,x}\|\|x\| = \|a\theta_{x,x}\|.\end{aligned}$$

So $\|ax\| \le \|a\theta_{x,x}\|$.

But

$$\|a\theta_{x,x}\| = \|\theta_{ax,x}\| \le \|ax\|\|x\| = \|ax\| .$$

So we have

$$\|a\theta_{x,x}\| = \|ax\| = \|a(\theta_{x,x})x\| .$$

Similarly

$$\|\alpha_t(a)\theta_{x,x}\| = \|\alpha_t(a)(\theta_{x,x})x\| .$$

Then

$$\begin{aligned}\|ax\| &= \|a(\theta_{x,x})x\| = \|a\theta_{x,x}\| = \|\alpha_t(a\theta_{x,x})\|\\ &= \|\alpha_t(a)\alpha_t(\theta_{x,x})\| = \|\alpha_t(a)\theta_{x,x}\| = \|\alpha_t(a)\theta_{x,x}x\|\\ &= \|\alpha_t(a)x\| = \|u_t(ax)\|.\end{aligned}$$

So u_t is well defined isometry on Ax. Since A includes $K(E)$ for every $z \in E$ we have $\theta_{z,x} \in K(E)$ hence $\theta_{z,x} \in Ax$. But $\theta_{z,x}(x) = <x, x> z = z$. So $z \in Ax$ and since $\overline{K(E)E} = E$ hence $\overline{[Ax]} = E$. Thus u_t can be extended to a unitary on E.

$$u_t^*(ax) = u_t^{-1}(ax) = \alpha_t^{-1}(a)x .$$

To justify group properties of u_t consider $u_0(ax) = \alpha_0(a)x = x$ and

$$<u_s\ u_t\ (ax), bx> = <u_t\ (ax), u_{-s}\ (bx)>$$

$$= <\alpha_t(a)x, \alpha_{-s}(b)x> = <\alpha_{-s}(b^*)\ \alpha_t(a)x, x>$$

$$= <\alpha_{-s}(b^*\ \alpha_{t+s})(a)x, x> = <u_{-s}(b^*\ \alpha_{t+s}(a)x), x>$$

$$= <b^*\ \alpha_{t+s}(a)x, u^*_{-s}(\theta_{x,x})x> = <b^*\ \alpha_{t+s}(a)x, \alpha_s(\theta_{x,x})x>$$

$$= <b^*\ \alpha_{t+s}(a)x, x> = <\alpha_{t+s}(a)x, bx> = <u_{t+s}(a)x, bx> .$$

for each a, $b \in A$. Since $\overline{[Ax]} = E$, then $u_{t+s} = u_t\ u_s$. Strong continuity of u_t follows by:

$$\|u_t(ax) - ax\| = \|(\alpha_t(a) - a)x\| \le \|\alpha_t(a) - a\|\|x\|$$

Therefore u_t is a C_0-group of unitaries on E. Finally for each $a,b \in A$

$$u_t a u_t^*(bx) = u_t a \alpha_t^{-1}(b)x$$
$$= \alpha_t\left(a\alpha_t^{-1}(b)\right)x$$
$$= \alpha_t(a)(bx)$$

It follows from the density of $[Ax]$ in E that $\alpha_t(a) = u_t a u_t^*$. □

Theorem 3. Let $A \subseteq L(E)$ be a C*-algebra including $K(E)$ and there is $x \in E$ with $<x,x>=1$ and α be an automorphism on A such that $\alpha(\theta_{x,x}) = \theta_{y,y}$ where $<y, y>=1$.

Then there is a unitary operator u in $L(E)$ such that $\alpha(a) = uau^*$, $(a \in A)$.

Proof. Define for each $a \in A$, $u(ax) = \alpha(a)y$. The proof of Theorem 2 we have:

$$\|u(ax)\| = \|\alpha(a)y\| = \|\alpha(a)\theta_{y,y}y\| = \|\alpha(a)\theta_{y,y}\|$$
$$= \|\alpha(a)\alpha(\theta_{x,x})\| = \|\alpha(a\theta_{x,x})\|\|a\theta_{x,x}\| = \|ax\|.$$

So u is well-defined isometry on Ax. Since $A \subseteq K(E)$, $\overline{[Ax]} = E$. So u is an invertible isometry on E with inverse $u^{-1}(ay) = \alpha^{-1}(a)x$. Hence u is a unitary operator. Now

$$uau^*(by) = ua\alpha^{-1}(b)x$$
$$= \alpha\left(a\alpha^{-1}(b)\right)y$$
$$= \alpha(a)(by) \qquad (b \in A)$$

which implies that $uau^* = \alpha(a)$ on Ay. The density of $\overline{[Ay]}$ in E completes the proof. □

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