Vol.4 No.3
Summer 1993 J. Sci. I. R. Iran

ON FINITENESS OF PRIME IDEALS IN NORMED RINGS

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Abstract

In a commutative Noetherian local complex normed algebra which is complete in its M-adic metric there are only finitely many closed prime ideals.

Introduction

The M-adic topology on a commutative ring R (with unity) is the one for which the open sets are unions of sets of the form $a + M^k$ ($a \in R;\ k = 0, 1, \ldots$) where M is an ideal of R. This topology makes R into a topological ring, and it is Hausdorff if and only if the intersection of powers of M is the zero ideal. Moreover, if R is Hausdorff then it is metrizable with the metric

$$d(x, y) = 2^{-k}$$

if and only if $x - y \in M^k$ and $x - y \in M^{k+1}$.

When R is the complex algebra of formal power series, there is also the topology of coefficientwise convergence on R, denoted by τ_c, which is the unique topology making R into a complete metrizable topological algebra [2; 5. 5]. Though τ_c is different from the M-adic topology on R where M is the maximal ideal generated by variables, nevertheless these topologies are closely connected; see [5; Theorem]. It is therefore conceivable that the M-adic topology should naturally arise in the study of Banach algebras B for which there exist unital monomorphisms

$$\mathbb{C}\,[[X_1, \ldots, X_n]] \rightarrow B.$$

This aspect of Banach algebra theory has been dealt with by G. R. Allan in his study of closed ideals in certain Banach algebras [1]. Since there are some interesting results in this area, perhaps not adequately known, this note is intended to make public knowledge an account of Allan's work on closed ideals by applying M-adic techniques; see the theorem below. For closed ideals of convolution algebras see [4].

Keywords: Normed algebra; Noetherian ring; M-adic topology; Closed ideal

Results

We begin by recalling that in a commutative Banach algebra every maximal ideal is closed [3; 11. 3(b)]. Such a result is not necessarily true for arbitrary normed algebras. However, we have:

Lemma. Suppose R is a commutative complex normed algebra with 1. Assume further that R is a local ring with the unique maximal ideal M. Then M is closed in R.

Proof. Let $\hat{R}$ be the norm completion of R. Since $\hat{R}$ is a commutative Banach algebra with 1, there is a character ψ on it. Now the restriction of ψ on R is a character on R and since $\ker(\psi | R)$ is a maximal ideal of R, it must be the unique maximal ideal M. So we have $M = \ker \psi \cap R$. Now $\ker \psi$ is closed in R since ψ is continuous and thus M is closed in R.

We can now state and prove the following:

Theorem. Suppose R is a commutative complex algebra with 1 which is also a Noetherian local ring with the unique maximal ideal M. Suppose further that R is complete in the M-adic metric and that $\|.\|$ is an algebra norm on R. Then R has only finitely many closed prime ideals with respect to this norm.

Proof. First we note that the set of all closed prime ideals of R is not empty since by Lemma, M is in this set. Now suppose $J_1, J_2, \ldots$ is a sequence of distinct closed prime ideals of R. We may now assume, without loss of generality, that for $i < J$ we have $J_i \not\subset J_j$. For, using the Noetherian condition, we let J_1 be a maximal element in $\{J_2, J_3, ...\}$, etc. Since J_j's are distinct, it thus follows that $J_k \not\subset J_r$ for $k<r$. So there exists an element $f_{k,r} \in J_k$ such that $f_{k,r} \notin J_r$. Define for any $k = 1, 2, ...,$

$$g_k = f_{1,k+1}\ f_{2,k+1} \cdots f_{k,k+1};$$

so we have $g_k \in M^k$. Now consider the sequence $\{\sum_{k=1}^{n} \lambda_k g_k\}_{n\geq 1}$ in R. It is easily checked that this sequence is Cauchy in the M-adic topology of R for any choice of $\lambda_k \in \mathbb{C}$, and so by the M-adic completeness of R it converges to a unique element of R. Set $f = \sum_{k\geq 1} \lambda_k g_k$ (M-adic convergence) where λ_k's are to be found. Take $\lambda_1 = 1$; and suppose $\lambda_1, \lambda_2, ..., \lambda_{n-1}$ are found for some $n \geq 2$. Let $\pi_n: R \to R/J_n$ be the canonical quotient mapping. Now $g_n = f_{1,n+1} f_{2,n+1} \cdots f_{n,n+1} \notin J_{n+1}$ since J_{n+1} is a prime ideal; so we have

$$\| \pi_{n+1}(g_n) \| > 0$$

Thus we can choose $\lambda_n \in C$ such that:

$$|\lambda_n| \|\pi_{n+1}(g_n)\| - \|\sum_{k=1}^{n-1} \lambda_k \pi_{n+1}(g_k)\|| > n.$$

This defines $\{\lambda_n\}_{n\geq 1}$ inductively. Now

$$f - \sum_{k=1}^{n} \lambda_k g_k = \sum_{k=n+1}^{\infty} \lambda_k g_k;$$

as R is a Noetherian local ring, from Theorem 9 in [6] page 262, we deduce that any ideal is closed in the M-adic topology. In particular, J_{n+1} is closed and since each $g_k \in J_{n+1}$ for $k = n+1, n+2, ...$, we have that

$$\sum_{k=n+1}^{\infty} \lambda_k g_k \in J_{n+1}$$

Thus

$$\| \pi_{n+1}(f) \| = \| \sum_{k=1}^{n} \lambda_k \pi_{n+1}(g_k) \| > n.$$

But $\| \pi_{n+1}(f) \| \leq \|f\|$ and so we have a contradiction. Thus, there are only finitely many closed ideals.

Remark. Suppose that R is a commutative Noetherian local complex algebra, which is complete in its M-adic metric and has infinitely many prime ideals. Then no algebra norm on R can possibly induce a topology equivalent with the M-adic topology.

Acknowledgements

The helpful comments of the referees are gratefully acknowledged.

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