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PERIODIC SOLUTIONS OF CERTAIN THREE DIMENSIONAL AUTONOMOUS SYSTEMS

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Abstract

There has been extensive work on the existence of periodic solutions for nonlinear second order autonomous differential equations, but little work regarding the third order problems. The popular Poincare-Bendixon theorem applies well to the former but not the latter (see [2] and [3]). We give a necessary condition for the existence of periodic solutions for the third order autonomous systems. This may become useful in further investigations. Our claims are proved and supported by certain examples.

Introduction

We consider real three dimensional autonomous systems,

$$\frac{dx}{dt} = P_3(x,y,z)$$
$$\frac{dy}{dt} = z + Q_3(x,y,z) \qquad (1)$$
$$\frac{dz}{dt} = -y + R_3(x,y,z),$$

where $P_3(x,y,z)$, $Q_3(x,y,z)$ and $R_3(x,y,z)$ are homogeneous polynomials of degree three having the following forms:

$$P_3(x,y,z)= a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3 + a_4xz^2 + a_5x^2z + a_6z^3 + a_7yz^2 + a_8y^2z + a_9xyz$$

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$$Q_3(x,y,z)= b_0x^3 + b_1x^2y + b_2xy^2 + b_3y^3 + b_4xz^2 + b_5x^2z + b_6z^3 + b_7yz^2 + b_8y^2z + b_9xyz$$

$$R_3(x,y,z)= c_0x^3 + c_1x^2y + c_2xy^2 + c_3y^3 + c_4xz^2 + c_5x^2z + c_6z^3 + c_7yz^2 + c_8y^2z + c_9xyz.$$

Obviously, the periodic solutions of the linear system are periodic with orbits which are circles with centers on the x-axis lying in a plane x = constant. Our approach is to assume that the full system has a periodic solution close to a circular orbit of the linear system in the plane $x= 0$. Therefore, we use the implicit function technique to establish the necessary condition in order for the solution of (1) to be closed in the neighborhood of the origin. The method which is used here is similar to Loud's [1].

Let $x(t,\xi), y(t,\xi), z(t,\xi)$ be that solution of (1) which has $x= 0, y= 0, z= \xi (\xi>0)$ at $t=0$. After a time approximately 2π, this solution will have made one cycle around the origin and will reach the point $(0,0,\xi)$ provided that the following equations are satisfied:

$$\begin{cases} F(\tau,\ \xi)= x(2\pi + \tau,\xi)= 0 \\ G(\tau,\ \xi)= y(2\pi + \tau,\xi)= 0 \\ H(\tau,\ \xi)= z(2\pi + \tau,\xi) - \xi = 0\ . \end{cases} \qquad (2)$$

If we solve $G(\tau, \xi)= 0$ for τ as a function of ξ for small ξ, say $\tau= \phi(\xi)$, we have that the time of return is $2\pi+ \phi(\xi)$. We then compute

$$J_1(\xi)= F(\phi(\xi), \xi), \quad J_2(\xi)= H(\phi(\xi), \xi).$$

We find that the position of return is $x = J_1(\xi)$, $y= 0$, $z= \xi+ J_2(\xi)$. Thus, the solution curve is closed if and only if $J_1(\xi)= J_2(\xi)= 0$.

We now proceed to investigate the asymptotic behavior of $J_1(\xi), J_2(\xi)$ as $\xi \rightarrow 0$. This will be done by computing the first three derivatives of $J_1(\xi)$, $J_2(\xi)$ at $\xi= 0$. Since $x(t,0) \equiv y(t,0) \equiv z(t,0) \equiv 0$, we have $F(0,0)= G(0,0)= H(0,0)= \phi(0)= J_1(0)= J_2(0)= 0$. The first derivatives of F, G and H at (0,0) are given by

$$F_\tau(0,0)= x'(2\pi,0)= 0 \quad F_\xi(0, 0)= x_\xi(2\pi, 0)$$

$$G_\tau(0,0)= y'(2\pi,0)= 0 \quad G_\xi(0, 0)= y_\xi(2\pi, 0)$$

$$H_\tau(0,0)= z'(2\pi,0)= 0 \quad H_\xi(0, 0)= z_\xi(2\pi, 0)-1.$$

Here, the prime symbol denotes differentiation with respect to t. The derivatives of x_ξ, y_ξ and z_ξ satisfy

$$x'_\xi= 3a_0x_\xi x^2 + \ldots + a_9(x_\xi yz + xy_\xi z + xyz_\xi)$$

$$y'_\xi= z_\xi + 3b_0x_\xi x^2 + \ldots + b_9(x_\xi yz + xy_\xi z + xyz_\xi)$$

$$z'_\xi= -y_\xi + 3c_0x_\xi x^2 + \ldots c_9(x_\xi yz + xy_\xi z + xyz_\xi)$$

with the initial conditions $x_\xi= 0$, $y_\xi= 0$ and $z_\xi= 1$ at $t= 0$. If we set $\xi= 0$ in the differential equations, they become

$$x'_\xi(t,0)= 0,\ y'_\xi(t,0)=z_\xi(t,0),\ z'_\xi(t,0)= -y_\xi(t,0).$$

With initial conditions, we obtain $x'_\xi(t,0)= 0$, $y'_\xi(t,0)= \sin t$, $z_\xi(t,0)= \cos t$. Hence,

$$F_\xi(0,0)= G_\xi(0,0)= H_\xi(0,0)= 0.$$

Therefore, to determine the behavior of $J_1(\xi)$ and $J_2(\xi)$ near $\xi=0$, it will be necessary to compute higher derivatives of F, G and H. The second derivatives of F, G and H at (0,0) are as follows:

$$F_{\tau\tau}(0,0)= x''(2\pi, 0)= 0 \quad F_{\tau\xi}(0,0)= x'_\xi(2\pi, 0)= 0 \quad F_{\xi\xi}(0,0)= x_{\xi\xi}(2\pi, 0)$$

$$G_{\tau\tau}(0,0)= y''(2\pi, 0)= 0 \quad G_{\tau\xi}(0,0)= y'_\xi(2\pi, 0) = 1 \quad G_{\xi\xi}(0, 0)= y_{\xi\xi}(2\pi, 0)$$

$$H_{\tau\tau}(0,0)= z''(2\pi, 0)= 0 \quad H_{\tau\xi}(0,0)= z'_\xi(2\pi, 0) = 0 \quad H_{\xi\xi}(0, 0)= z_{\xi\xi}(2\pi, 0).$$

Obviously, the derivatives $x_{\xi\xi}$, $y_{\xi\xi}$ and $z_{\xi\xi}$ satisfy

$$x'_{\xi\xi}= 3a_0(x_{\xi\xi}x^2+2x^2_\xi x) + \ldots + a_9(x_{\xi\xi}yz + xy_{\xi\xi}z + xyz_{\xi\xi} +2x_\xi y_\xi z + 2xy_\xi z_\xi + 2zx_\xi y_\xi)$$

$$y'_{\xi\xi}= z_{\xi\xi} + 3b_0(x_{\xi\xi}x^2+2x^2_\xi x) + \ldots + b_9(x_{\xi\xi}yz + xy_{\xi\xi}z + xyz_{\xi\xi} + 2x_\xi y_\xi z + 2xy_\xi z_\xi + 2zx_\xi y_\xi)$$

$$z'_{\xi\xi}= -y_{\xi\xi} + 3c_0(x_{\xi\xi}x^2+2x^2_\xi x) + \ldots + c_9(x_{\xi\xi}yz + xy_{\xi\xi}z + xyz_{\xi\xi} + 2x_\xi y_\xi z + 2xy_\xi z_\xi + 2zx_\xi y_\xi).$$

The initial values of $x_{\xi\xi}$, $y_{\xi\xi}$ and $z_{\xi\xi}$ at $t= 0$ are zero. Setting $\xi= 0$ in the above equation, we obtain

$$x'_{\xi\xi}= 0,\ y'_{\xi\xi}= z_{\xi\xi},\ z'_{\xi\xi}= -y_{\xi\xi}. \tag{3}$$

The solution of this system with the given initial conditions is:

$$x_{\xi\xi}(t, 0)= y_{\xi\xi}(t, 0)= z_{\xi\xi}(t,0) \equiv 0.$$

From this, we conclude that $F_{\xi\xi}(0,0)= G_{\xi\xi}(0,0)= H_{\xi\xi}(0,0)= 0$. The third derivatives of F, G and H at (0,0) are computed in a similar manner and we have:

$$F_{\tau\tau\tau}(0,0)= x'''(2\pi,0)= 0 \quad G_{\tau\tau\tau}(0,0)= y'''(2\pi,0)= 0 \quad H_{\tau\tau\tau}(0,0)= z'''(2\pi, 0)= 0$$

$$F_{\tau\tau\xi}(0,0)= x''_\xi(2\pi,0)= 0 \quad G_{\tau\tau\xi}(0,0)= y''_\xi(2\pi,0)= 0 \quad H_{\tau\tau\xi}(0,0)= z''_\xi(2\pi, 0)= -1$$

$$F_{\tau\xi\xi}(0,0)= x'_{\xi\xi}(2\pi,0)= 0 \quad G_{\tau\xi\xi}(0,0)= y'_{\xi\xi}(2\pi,0)= 0 \quad H_{\tau\xi\xi}(0,0)= z'_{\xi\xi}(2\pi, 0)= 0$$

$$F_{\xi\xi\xi}(0,0)= x_{\xi\xi\xi}(2\pi,0)= 0 \quad G_{\xi\xi\xi}(0,0)= y_{\xi\xi\xi}(2\pi,0)= 0 \quad H_{\xi\xi\xi}(0,0)= z_{\xi\xi\xi}(2\pi, 0).$$

Now, the derivatives of $x_{\xi\xi\xi}$, $y_{\xi\xi\xi}$ and $z_{\xi\xi\xi}$ satisfy

$$x'_{\xi\xi\xi}= 6a_0x^3_\xi + 6a_1x^2_\xi y_\xi + 6a_2x_\xi y^2_\xi + 6a_3y^3_\xi + 6a_4x_\xi z^2_x + 6a_5x^2_\xi z_\xi + 6a_6z^3_\xi + 6a_7y_\xi z^2_\xi + 6a_8y^2_\xi z_\xi + 6a_9x_\xi y_\xi z_\xi + \ldots$$

$$y'_{\xi\xi\xi}= z_{\xi\xi\xi} + 6b_0x^3_\xi + 6b_1x^2_\xi y_\xi + 6b_2x_\xi y^2_\xi + 6b_3y^3_\xi + 6b_4x_\xi z^2_\xi + 6b_5x^2_\xi z_\xi + 6b_6z^3_\xi + 6b_7y_\xi z^2_\xi + 6b_8y^2_\xi z_\xi + 6b_9x_\xi y_\xi z_\xi + \ldots$$

$$z'_{\xi\xi\xi}= -y_{\xi\xi\xi} + 6c_0x^3_\xi + 6c_1x^2_\xi y_\xi + 6c_2x_\xi y^2_\xi + 6c_3y^3_\xi + 6c_4x_\xi z^2_\xi + 6c_5x^2_\xi z_\xi + 6c_6z^3_\xi + 6c_7y_\xi z^2_\xi + 6c_8y^2_\xi z_\xi + 6c_9x_\xi y_\xi z_\xi + \ldots$$

The initial values of $x_{\xi\xi\xi}$, $y_{\xi\xi\xi}$ and $z_{\xi\xi\xi}$ at $t= 0$ are zero.

Setting $\xi = 0$ in the above equations, we shall obtain

$$x'_{\xi\xi\xi} = 6a_3\sin^3 t + 6a_6\cos^3 t + 6a_7\sin t\cos^2 t + 6a_8\sin^2 t\cos t$$

$$y'_{\xi\xi\xi} = z_{\xi\xi\xi} + 6b_3\sin^3 t + 6b_6\cos^3 t + 6b_7\sin t\cos^2 t + 6b_8\sin^2 t\cos t$$

$$z'_{\xi\xi\xi} = -y_{\xi\xi\xi} + 6c_3\sin^3 t + 6c_6\cos^3 t + 6c_7\sin t\cos^2 t + 6c_8\sin^2 t\cos t$$

or

$$x'_{\xi\xi\xi} = \frac{3}{2}(3a_3 + a_7)\sin t + \frac{3}{2}(3a_6 + a_8)\cos t + \frac{3}{2}(a_7 - a_3)\sin 3t + \frac{3}{2}(a_6 - a_8)\cos 3t$$

$$y''_{\xi\xi\xi} + y_{\xi\xi\xi} = \frac{3}{2}[3(c_3 - b_6) + (c_7 - b_8)]\sin t + \frac{3}{2}[3(c_6 + b_3) + c_8 + b_7]\cos t + \frac{3}{2}[c_7 - c_3 + 3(b_8 - b_6)]\sin 3t + \frac{3}{2}[c_6 - c_8 + 3(b_7 - b_3)]\cos 3t$$

$$z'''_{\xi\xi\xi} + z_{\xi\xi\xi} = -\frac{3}{2}[3(b_3 + c_6) + (b_7 + c_8)]\sin t + \frac{3}{2}[3(c_3 - b_6) + c_7 - b_8]\cos t - \frac{3}{2}[b_7 - b_3 - 3(c_6 - c_8)]\sin 3t + \frac{3}{2}[3(c_7 - c_3) - (b_6 - b_8)]\cos 3t.$$

Therefore,

$$x_{\xi\xi\xi} = (4a_3 + 2a_7) - \frac{3}{2}(3a_3 + a_7)\cos t + \frac{3}{2}(3a_6 + a_8)\sin t - \frac{1}{2}(a_7 - a_3)\cos 3t + \frac{1}{2}(a_6 - a_8)\sin 3t$$

$$y_{\xi\xi\xi} = \frac{3t}{4}\{[3(c_6 + b_3) + c_8 + b_7]\sin t - [3(c_3 - b_6) + c_7 - b_8]\cos t\} - \frac{3}{16}[c_7 - c_3 - 3(b_6 - b_8)]\sin 3t - \frac{3}{16}[c_6 + b_7 - c_8 - b_3]\cos 3t + \frac{3}{16}[c_6 + b_7 - c_8 - b_3]\cos t$$

$$z'_{\xi\xi\xi\xi} = \frac{3t}{4}\{[3(c_3 - b_6) + c_7 - b_8]\sin t + [3(b_3 + c_6) + b_7 + c_8]\cos t\} + \frac{3}{16}[b_7 - b_3 + 3(c_6 - c_8)]\sin 3t - \frac{3}{16}[3(c_7 - c_3) - (b_6 - b_8)]\cos 3t + \frac{3}{16}[3(c_7 - c_3) - (b_6 - b_8)]\cos t.$$

Hence,

$$F_{\xi\xi\xi}(0, 0) = x_{\xi\xi\xi}(2\pi, 0) = 0$$

$$G_{\xi\xi\xi}(0, 0) = y_{\xi\xi\xi}(2\pi, 0) = \frac{3\pi}{2}[3(b_6 - c_3) + (b_8 - c_7)]$$

$$H_{\xi\xi\xi}(0, 0) = z_{\xi\xi\xi}(2\pi, 0) = \frac{3\pi}{2}[3(b_3 + c_6) + (b_7 + c_8)].$$

The fourth derivatives of $F(\tau, \xi)$ are given as below:

$F_{\tau\tau\tau\tau}(0, 0) = x''''(2\pi, 0) = 0$, $F_{\tau\tau\tau\xi}(0, 0) = x'''_{\xi}(2\pi, 0) = 0$, $F_{\tau\tau\xi\xi}(0, 0) = x''_{\xi\xi}(2\pi, 0) = 0$,

$F_{\tau\xi\xi\xi}(0, 0) = x'_{\xi\xi\xi}(2\pi, 0) = 6a_6$, $F_{\xi\xi\xi\xi}(0, 0) = x_{\xi\xi\xi\xi}(2\pi, 0) = 0$.

We have now enough information to make statements about the functions $\phi(\xi)$, $J_1(\xi)$ and $J_2(\xi)$. From the above, we realize that $F(\tau, \xi)$, $G(\tau, \xi)$ and $H(\tau, \xi)$ have the following expansions near (0, 0):

$$F(\tau, \xi) = \frac{3}{2}a_6\tau\xi^2 + \ldots$$

$$G(\tau, \xi) = \tau\xi + \frac{1}{6}G_{\xi\xi\xi}(0, 0)\xi^3 + \ldots$$

$$H(\tau, \xi) = -\frac{1}{2}\tau^2\xi + \frac{1}{6}H_{\xi\xi\xi}(0, 0)\xi^3 + \ldots$$

where the missing parts are of the order 4 (the combinations of τ and ξ). Solving the second equation for τ as a function of ξ, we find

$$\tau = \phi(\xi) = -\frac{1}{6}G_{\xi\xi\xi}(0, 0)\xi^2 + \ldots = O(\xi^2)$$

for small ξ. Substituting this expansion for $F(\tau, \xi)$ and $H(\tau, \xi)$, we obtain

$$J_1(\xi) = F(\phi(\xi), \xi) = O(\xi^4)$$

$$J_2(\xi) = H(\phi(\xi), \xi) = \frac{1}{6}H_{\xi\xi\xi}(0, 0)\xi^3 + O(\xi^4).$$

Therorem 1. A necessary condition that the system (1) have a closed curve solution in the neighborhood of the origin is that

$$3(b_3 + c_6) + (b_7 + c_8) = 0. \qquad (4)$$

Moreover, the necessary condition for having a periodic solution of period 2π is:

$$3(b_6 - c_3) + (b_8 - c_7) = 0.$$

Proof. The condition (4) is equivalent to $H_{\xi\xi\xi}(0,0)=0$. If $H_{\xi\xi\xi}(0,0)\neq 0$, it is clear from the formula for $J_2(\xi)$ that $J_2(\xi)\neq 0$ for small $\xi\neq 0$. Hence, there will be no closed curve solution for (1) in some neighborhood of the origin. Now, if we assume that (1) has a periodic solution, then the period of the periodic solution is given by $T=2\pi+\phi(\xi)$, where $\phi(\xi)$ is determined by

$$\tau=\phi(\xi)\Leftrightarrow G(\tau,\xi)=0.$$

Furthermore, for small values of ξ we have,

$$\phi(\xi)=-\frac{1}{6}G_{\xi\xi\xi}(0,0)\xi^2+O(\xi^3).$$

Hence, the asymptotic behavior of the period of the periodic solution $x(t,\xi)$, $y(t,\xi)$ and $z(t,\xi)$ as $\xi\to 0$ is given by

$$T(\xi)=2\pi-\frac{\pi}{4}[3(b_6-c_3)+(b_8-c_7)]\xi^2+O(\xi^3). \quad (5)$$

Thus, a necessary condition in order for the solution curve to be periodic of period 2π is:

$$3(b_6-c_3)+(b_8-c_7)=0. \square \quad (6)$$

Periodic solutions for nonlinear third order differential equations have not been investigated extensively. In the next two examples, we consider two such problems and verify the necessities outlined in Theorem 1.

Example 1. Consider

$$(A_1)\ x'''+x'+a_1x'x^2+a_3x'x'^2+(a_1+a_3)xx'x''=0,$$

which has a periodic solution of period 2π, $x=\sin t$. If we write (A_1) as a system with $x'=-y$, $y'=z$, we obtain:

$$x'=-y,\ y'=z,\ z'=-y-a_1yx^2-a_3yz^2+(a_1+a_3)xyz.$$

This is (1) with $b_3=c_6=b_7=c_8=0$ and $b_6=c_3=b_8=c_7=0$; i.e., the necessary conditions (4) and (6) are satisfied.

Example 2. Consider

$$(A_2)\ \frac{d^3x}{dt^3}+(1+3x^2)\frac{dx}{dt}-3x\left(\frac{dx}{dt}\right)^2+x^3=0.$$

Again if we write (A2) as a system with $x'=-y$, $y'=z$, we find

$$\frac{dx}{dt}=-y,\ \frac{dy}{dt}=z,\ \frac{dz}{dt}=-y-3x^2y-3xy^2+x^3.$$

Taking the transformation $x=x+z$ (or $x=x-z$), we find: find:

$$\frac{dx}{dt}=-x^3+3x^2y+3xy^2-3x^2z-3xz^2+3zy^2+3z^2y-z^3+6xyz$$

$$\frac{dy}{dt}=z$$

$$\frac{dz}{dt}=-y+x^3-3x^2y-3xy^2+3xz^2+3x^2z+z^3-3z^2y-3zy^2-6xyz.$$

This is (1) with all coefficients $b_i=0$, $i=0,1,\ldots,9$ and $c_6=1$, $c_8=-3$. The problem (A_2) has a closed curve solution and the necessary condition (4) is satisfied. In fact, the problem has a periodic solution provided that ξ is sufficiently small. The period of the solution is governed by:

$$T(\xi)=2\pi-\frac{\pi}{4}(-3)\xi^2+O(\xi^3)=2\pi+\frac{3\pi}{4}\xi 2+O(\xi^3).$$

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