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ON MULTIPHASE ALGORITHM FOR SINGLE VARIABLE EQUATION USING NEWTON'S CORRECTION METHOD

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Abstract

This paper brings to light a method based on Multiphase algorithm for single variable equation using Newton's correction. Newton's method is derived through the logarithmic differentiation of polynomial equation. A correction term which enhances the high speed of convergence is hereby introduced. A translation of Newton's method to Total Step and Single Step Methods (T. S. M and S. S. M) respectively, forms the peak of discussion. Our method, so derived, is also discussed in the light of numerical evidence.

1. Introduction

A mapping Φ of a subset of complex sequences $\mathbb{C}$ into $\mathbb{C}$ is called a sequence transformation (or simply summability method), for such a mapping $\Phi(z) = z$ if $z_n \to z$. When this holds for all convergent sequences, Φ is said to be regular. We say that Φ is accelerative for z if one can find $z_n \to z$ and

$$z_n - z = O(z_n - z)$$

Often times, numerical analysts are interested in mappings that are accelerative, precisely and strongly accelerative in the sense that $z_n \to z$ as fast as possible for as large a class of convergent sequences.

In this paper, we are concerned with the problem of approximating zeros of nonlinear equation.

$$p(z) = 0 \quad (1.1)$$

where p is assumed to be continuous in its domain of definition. We also assumed that P is monotone of an interval $z^{(0)} \in \mathbb{C}$. By using interval arithmetic, it is possible to compute a zero of P in the interval [a,b]. It is also assumed that the derivative of p (z) where $z \in z^{(0)}$ has an interval extension $p'(z) \in \mathbb{C}, z \subset z^{(0)}$ with the properties

$p'(z) \in p'(z)$ for all

$$z \in z \subset z^{(0)} \quad (1.2)$$

$$p'(z) \subseteq p'(z) \quad \text{if } z \subseteq z^{(0)} \quad (1.3)$$

$$d(p'(z) \subseteq d(z) \text{ for all } z \subseteq z(o) \quad (1.4)$$

Observe that condition (1.4) defines Lipschtz continuity of the point derivative $p'(z)$. It is worthwhile to note that the interval method we wish to discuss in this paper will break down if zero of polynomial p does not exist in an interval under consideration.

This paper is arranged as follows: In section 2, we have presented a family of Weierstrass' methods as a prelude to discussing Newton's method. Thus, our method is in the same spirit as those works. Section 3 discusses the so called Total Step Method (T.S.M) and Single Step Method (S.S.M) of Newton. In section 4, we have presented some numerical results to support our claims.

Keywords: Newton's method; Weierstrass' method; Newton's correction; Logarithmic differentiation of polynomial; Interval arithmetics

2. The Multiphase Algorithm of Newton's Method

For a polynomial of degree n we define

$p(z) = z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = (z - \xi_2)^{\mu_1} \dots (z - \xi_n)^{\mu_k}$

where

$\mu_1 + \mu_2 + \dots + \mu_k = n$

and repeated zeros are counted according to their multiplicities so that the number of zeros are exact. In case of simple zeros, we have $1 = 1 = \mu_1 = \mu_2 = \mu_k$ and $k = n$.

Given n pairwise distinct approximants $(Z_1, Z_2, \dots, Z_n) \in \mathbb{C}^n$ for the n pairwise distinct zeros$(\xi_1, \xi_2, \dots \xi_n) \in \mathbb{C}^n$ of a monic polynomial p of degree $n > 3$, one iteration step of Durand Kerner's method (see [3]) reads:

$$Z_i^{(m+1)} = Z_i^{(m)} - W_i^{(m)} \tag{2.1}$$

$(i = 1, 2, 3, \dots, n), m = 0, 1, \dots$

where

$$W_i^{(m)} = \frac{P(Z_i^{(m)})}{\prod_{\substack{j=1\\ j=1i}}^{n} (Z_i^{(m)} - Z_j^{(m)})} \tag{2.2}$$

is called Weierstrass' correction. Method (2.1) has been modified in many ways by different authors (see [5, 6]):

$$Z_i^{(m+1)} = Z_i^{(m)} - \frac{W_i^{(m)}}{1 - \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{W_j^{(m)}}{Z_i^{(m)} - Z_j^{(m)} + W_j^{(m)}}} \tag{2.3}$$

Method (2.3) is the popular Weierstrass method (often called Nourine's method) which has a fourth order of convergence. (See [3] for more details).

Many methods exist for deriving Newton's method which is based on fixed point principle. In this paper, we shall adopt the approach of logarithmic differentation of polynomial

$$P(Z) = (Z_i - \xi_j)^{\mu_j} \prod_{\substack{j=1\\ j=1}}^{n} (Z_i - \xi_j)^{\mu_j} \tag{2.4}$$

where it is supposed that $(\xi_1, \xi_2, \dots \xi_n)$ are approximate zeros to $Z_1, Z_2, \dots Z_n$. Taking $\log_e$ of both sides of (2.4), we have

$$Log_e P(Z) = \mu_i Log_e (Z_i - \xi_i) + \mu_j Log_e \sum_{\substack{j=1\\ j=i}}^{n} (Z_i - \xi_i) \tag{2.5}$$

If we differentiate Equation (2.5) with respect to Z_i, we obtain

$$\frac{P'(Z_i)}{P(Z_i)} = \frac{\mu_i}{Z_i - \xi_i} + \sum_{\substack{j=1\\ j=1}}^{n} \frac{\mu_j}{Z_i - \xi_i}$$

Rearranging terms, we get

$$\frac{\mu_i}{Z_i - \xi_i} = \frac{P'(Z_i)}{P(Z_i)} = + \sum_{\substack{j=1\\ j=1}}^{n} \frac{\mu_j}{Z_i - \xi_i} \tag{2.6}$$

Taking inverse of both sides of Equation (2.6) and rearranging common terms, we have

$$Z_i = \xi_i = \frac{\mu_i}{\left[\frac{P'(Z_i)}{P(Z_i)} - \sum_{\substack{j=1\\ j=1}}^{n} \frac{\mu_j}{Z_i - \xi_i}\right]} \tag{2.7}$$

which implies that

$$\xi_i = Z_i - \frac{\mu_i}{\left[\frac{P'(Z_i)}{P(Z_i)} - \sum_{\substack{j=1\\ j=1}}^{n} \frac{\mu_j}{Z_i - \xi_i}\right]} \tag{2.8}$$

If we divide the Weight function of (2.8) by $P(Z_i)/P'(Z_i)$, we obtain

$$\xi_i = Z_i - \frac{\mu_i P(Z_i) / P'(Z_i)}{\left[1 - \frac{P(Z_i)}{P'(Z_i)} - \sum_{\substack{j=1\\ j=1}}^{n} \frac{\mu_j}{Z_i - \xi_i}\right]} \tag{2.9}$$

If ξ is a reasonable approximation to $\hat{Z}$ we have

$$\hat{Z}_i = \frac{\mu_i P(Z_i) / P'(Z_i)}{1 - \frac{P(Z_i)}{P'(Z_i)} - \sum_{\substack{j=1\\ j=1}}^{n} \frac{\mu_j}{Z_i - \xi_i}} \tag{2.10}$$

Our method of Equation (2.10) has a cubic order of convergence.

Define $Z_j = \xi_j - N_j$ where

$N_j = P(Z_j) / P'(Z_j)$

Then by implication $\xi_j = Z_j - N_j$ and hence, our method (2.10) now assumes the general form

$$\hat{Z}_i = Z_i - \frac{\mu_i P(Z_i) / P'(Z_i)}{\left[1 - \frac{P(Z_i)}{P'(Z_i)} - \sum_{\substack{j=1\\ j=1}}^{n} \frac{\mu_j}{Z_i - Z_j + N_j}\right]} \tag{2.11}$$

From the knowledge of R-order of convergence using the approach of Ortega and Rheinboldt [7], our method in (2.11) has beem increased from cubic order to focus order method owing to the introduced term N_j, called Newton's correction. We now bring to focus the relationship existing between methods (2.3) and (2.11) in the following setting:

$$\frac{P(Z_i)}{P'(Z_i)}\sum_{\substack{j=1\\ j=1}}^{n}\frac{1}{Z_i-Z_j+N_j}=\frac{p(z_i)}{\prod_{\substack{j=1\\ J=1}}^{n}(Z_i-Z_j+W_j)} \quad (2.12)$$

By carefully translating the term

$\dfrac{P(Z_i)}{\prod_{\substack{j=1\\ J=1}}^{n}(Z_i-Z_j+W_j)}$ in line with Lagrange's interpolation of polynomial (see [2 and 4]), the desired relationship existing in the equality of Equation (2.12) holds.

We estimate, (in some sense) the radius of the approximate zeros of polynomial P by using the Breass and Hadeler's disk [2] given by:

$$|Z - Z_i| \le n\,(N_j(Z)) \quad (2.13)$$

From the well known techniques of G-erschgorine Circle theorem for the inclusion of polynomial zeros, we may adopt the some techniques used by Petkovic et al [9] which was applied to Weierstrass' method (2.3) to include all zero sets of polynomial P. As a result, we now state Theorems 1 and 2 to press home our discussions.

Theorem 1 [9]

For $P \in \{1,2, ..., n\}$ and $\xi \in ¢$, let r be a positive number bounded by

$$\max_{j=1,2,\dots,n}\,(|Z_j - N_j - x| - |N_j|) < r < \min_{j=P+1,\dots,n}\,(|Z_j - N_j - x| N_j|) \quad (2.14)$$

such that

$$1 > h(r) = \sum_{j=P+1}^{n}\frac{|N_j|}{|Z_j - N_j - \xi| + |N_j| - r} \ge 0$$

then there are exactly P-zeros in the open disk with center ξ and radius r. The conditon in which $P = n$ and $h(r) \le 1$ is also discussed in [9].

Theorem 2 [9]

Let $Z_1, ... Z_n \in ¢ / \{Z_1, Z_2, ... Z_n\}$ be pairwise distinct and set

$$\delta_i = |N_j| \max_j |Z_j - \xi|^{-1},\ (J = 1,2, ..., n)$$

$$G_i = \sum_{\substack{j=1\\ j=1}}^{n}\frac{|N_j|}{Z_j - \xi|},\ i \in \{1, 2, ...n\}$$

if $\sqrt{1+\delta_i} > \sqrt{\delta_i} + \sqrt{G_i}$, for any $I = 1, 2, ..., n$, then the disk with centre Z_i - Mid (N_i) and radius

$$|N_i|\left[1 - \frac{2(1-2\delta_i - G_i)}{1-\delta_1 - 2G_i + \sqrt{(1-\delta_i - 2G_i)^2 + 4\delta_i(1-2\delta_i - G)^2}}\right] + rad(N_i$$

includes all the zeros sets of P.

The proof of Theorems 1 and 2 can be found in [9]. It can be shown that the disk $Z_i - Z_j + N_j$ of our method (2.11) does not contain the origin by carefully adopting the approach used in [8] for Weierstrass' method. As a consequence, we now state Theorem 3.

Theorem 3 [8]

Let $r = \max \text{rad}(Z_j)$, $j = 1, 2, ..., n$.

$$d = \min_{\substack{i,j=1,2,\dots,n\\ i=j}} |\,mid(Z_i) - mid(Z_i)|$$

If $d/r \ge 4n$ and $\xi_j \in Z_j = 1,2, ...,n$, then the inversions in our method (2.11) exist (i.e) $O \notin Z_i - Z_i + N_j$ for all $i, j \in \{1,2,...,n\}$.

In the next section we shall examine the convergence speed of our method in (2.11) relative to the applications in Total Step Method (T.S.M) and Single Step Method (S.S.M).

3. Convergence Speed of Newton's Method

We set out to analyse the R-order of convergence of Newton's Method of equation (2.11). The following notation shall be adopted.

Let

$h_i^{(k)} = 1 \le \max |Z_i^{(k)} - r_i|,\ i = 1, 2, ..., n\ k = 0, 1, ...$

I = iterative process with limit point Z*

Definition 1

The quantity

$$O_R(I,Z^*) = \begin{bmatrix} \infty \ if R_P(I, Z^*) = 0 \ for\ all\ p\ [1, \infty] \\ inf\{P \in [1, \infty] \mid R_P(I, Z^*) = 1\} \end{bmatrix}$$

Otherwise, it is called the R-order of I at Z*.

Definition 2

$R_p(I, Z^*)$ is called the *R*-factor of *I* at *Z** and is defined by

$$R_p(I, Z^*) = \sup\{R_p\{Z^{(k)}\} \mid \{Z^{(k)}\} \in C(I, Z^*), I < P < \infty\}$$

$$\text{where } R_p(Z^{(k)}) = \begin{cases} \limsup_{k\to\infty} |Z^{(k)} - Z^*|^{1/k} & if\ p = 1 \\ \limsup_{j\to\infty} |^{(k)} - Z^*|^{1/p^k} & if\ p > 1 \end{cases}$$

and C (I, Z*) is the set of all sequences generated by I and converging to Z*.

Definition 3

Let 1_1 and 1_2 be two iterative processes with the same limit Z^*. We say that, 1_1 is R-faster than 1^2 at Z^* if there is a $p[1, \infty]$ such that $R_p(1, Z^*) < R_p(1_2, Z^*)$. Furthermore, if $O_R(1_1, Z^*) > O_R(1_2, Z^*)$, then 1_1 is R-faster than 1_2 at z^*. With all these preambles, we introduce the T. S. M and S. S. M of method (2.11) for the case of simple zeros. (T. S. M).

$$Z_i^{(m+1)} = Z_i^{(m)} \frac{P(Z_i^{(m)} / P'(Z_i^{(m)})}{1 - \frac{P(Z_i^{(m)}}{P'(Z_i^{(m)}} \sum_{\substack{j=1 \\ j=1}}^{n} \frac{1}{Z_i^{(m)} - Z_i^{(m)} - N_j^{(m)}}} \quad (3.1)$$

(S.S.M)

$$Z_i^{(m+1)} = Z_i^{(m)} \frac{P(Z_i^{(m)} / P'(Z_i^{(m)})}{1 - \frac{P(Z_i^{(m)}}{P'(Z_i^{(m)}} \sum_{\substack{j=1 \\ j=1}}^{i-1} \frac{1}{Z_i^{(m)} - Z_i^{(m)} - N_j^{(m)}} \sum_{j=i+1}^{n} \frac{1}{Z_i^{(m)} - Z_i^{(m)} - N_j^{(m)}}}$$

It can be proven that method (3.2) converges faster than method (3.1) for instance, see Alefeld and Herzberger [1] for more. One thing about methods (3.1) and (3.2) is that they do not exhibit "zero divide in the course of the iteration cycle. This is possible because the term $P(Z_i^{(m)} / P'(Z_i^{(m)})$ does exhibit "zero divide" in the course of the iteration cycle.

4. Numerical Results

We apply our method (2.11) to the following polynomial problem, using complex interval arithmetic.

(1) $z^7 - 28z^6 + 323z^5 - 1960z^4 + 6769z^3 + 13068z^2 - 5040 = 0$

The exact roots are 1, 2, 3, 5, 6 and 7.

Using a test for the exclusion region to the polynomial problem above, it is found that the initial disks are in the intervals:

$$z_1^{(0)} = [0.8, 1.2]$$
$$z_2^{(0)} = [1.8, 2.3]$$
$$z_3^{(0)} = [2.6, 3.1]$$
$$z_4^{(0)} = [3.8, 4.2]$$
$$z_6^{(0)} = [5.8, 6.2]$$
$$z_7^{(0)} = [6.9, 7.3]$$

NITIAL ROOTS			
0.800000000	0.000000000	1.200000000	0.000000000
1.800000000	0.000000000	2.300000000	0.000000000
2.600000000	0.000000000	3.100000000	0.000000000
3.800000000	0.000000000	4.200000000	0.000000000
4.700000000	0.000000000	5.200000000	0.000000000
5.800000000	0.000000000	6.200000000	0.000000000
5.800000000	0.000000000	7.300000000	0.000000000
ITERATION 1			
1.000000000	0.000000000	1.000000000	0.000000000
2.049987700	0.000000000	2.049990000	0.000000000
2.849988000	0.000000000	2.849999000	0.000000000
4.000000000	0.000000000	4.200000000	0.000000000
4.94998000	0.000000000	4.949998000	0.000000000
6.000000000	0.000000000	6.000000000	0.000000000
6.549936000	0.000000000	6.549949900	0.000000000
ITERATION 2			
1.000000000	0.000000000	1.000000000	0.000000000
2.049977000	0.000000000	2.049977000	0.000000000
2.839977000	0.000000000	2.849977000	0.000000000
4.94999600	0.000000000	4.000000000	0.000000000
4.94998000	0.000000000	4.949996000	0.000000000
6.549886000	0.000000000	6.549886000	0.000000000

From the table above, it can be seen that convergence was achieved after two iterations circles. It can also be observed that the appropriate results obtained are also close to the true values of the solution to the polynomial problems.

Our method (2.11) was programmed to accommodate polynomial with complex roots. Thus, the method works well for the solutions of roots of polynomial, be it complex or real roots. The technique for the application of our method (2.11) in interval arithmetic is simple, which we expressed. Thus,

$$Z_i^{(m+1)} = m(Z_i^{(m)}) - \frac{Pm(Z_i^{(m)}) / P'(Z_i^{(m)})}{1 - \sum_{j=1}^{n} \frac{Pm(Z_i^{(m)}) / P'm(Z_i^{(m)})}{Z_i^{(m)} - Z_j^{(m)} + N_i^{(m)}}} \cap Z_i^{(m)} \quad (4.1)$$

were

$$M[a,b] = \frac{[a+b]}{2}$$

and $P'(Z_i^{(m)})$ has an interval extension of $(Z_i^{(m)})$

The advantage which our interval method (4.1) has over the real floating point method, (classical method) is that it provide tighter bounds for both lower and upper values of the sought results.

It is also on record that the interval method is self validating and one does not need to start calculating associated errors such as local truncation errors, propagating errors, etc. that are inherent with real floating point arithmetics. Thus, interval arithmetics is gaining ground in the field of scientific computings worldwide. The only disadvantage which interval arithmetic has is the problem of "wrapping effects" caused by the over-estimation of intervals which can be overcome in some cases.

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