VISCOSITY COEFFICIENTS OF THE A-PHASE OF SUPERFLUID He AT LOW TEMPERATURES

M.A. Shahzamanian

Department of Physics, Faculty of Sciences, University of Isfahan, 81744, Islamic Republic of Iran

Abstract

It has been shown that the Boltzmann equation for the viscosity of the A-phase of superfluid ^3He at low temperatures has the same form as for a normal Fermi liquid. The viscosity coefficients A, B, C, D and E are derived exactly. The λ_2 parameter plays an important role in determining the temperature dependence of the viscosity coefficients.

Introduction

Analytical evaluation of the shear viscosities of superfluid phases of ^3He has been investigated by many authors. These investigations have been concentrated mostly on the B-phase of superfluid ^3He, which is believed to be an isotropic BW state. However, even in this isotropic state, the collision integrals in the Boltzmann equation are very complicated and an exact solution for the Boltzmann equation cannot be found in the whole range of temperatures, except in those near transition temperature, T_C, and at the low temperature limit, $T\rightarrow 0$ [1,2].

The Boltzmann equation has been solved by approximate methods for a whole range of temperatures in the B-phase [3-5] which may be mainly generated in two ways: the first is by using the variational solution of the kinetic equation which one can use to find an approximate solution of the exact collision integral once the exact solutions in the limiting cases are known [4,5]. Using the second method, one may construct an approximation of the collision integral that allows an exact solution [4,6]; the simplest one is to equate the collision integral to the relaxation rate of the quasiparticle [7].

The Boltzmann equation contains the streaming terms and the collision integrals. In a normal Fermi liquid at low temperatures the only important collision process is the scattering of pairs of quasiparticles, but in a superfluid state the quasiparticle number is not conserved, so one also has to take into account decay processes in which a single quasiparticle decays into three, and the inverse processes, in which three quasiparticles coalesce to form one. For a normal Fermi liquid, the Boltzmann equation has been solved exactly by Sykes and Brooker [8] and Hojgaard Jensen *et al.* [9]. This equation has also been solved exactly close to T_C in the A and B-phases of superfluid ^3He by Bhattacharyya *et al.* [1]. They showed that the Boltzmann equation can be solved exactly by treating the difference between the collision terms for the normal and superfluid states as a perturbation. Bhattacharyya *et al.* [1] found that the shear viscosities drop as $(T_C-T)^{1/2}$ for temperatures just below T_C and the coefficients of $(T_C-T)^{1/2}$ are expressed as a function of normal state properties. Pethick *et al.* [2] solved the Boltzmann equation exactly at the low temperature limit for the BW state. They showed that the shear viscosity tends to a constant, and the two-quasiparticle scattering process contributes to the collision integral. The last result has been obtained by the argument that the total rate of coalescence and decay processes are less important by a factor $\sim e^{-\Delta/K_BT}$ than the two-quasiparticle scattering.

In this paper, we solve the Boltzmann equation exactly for the ABM state at the low temperatures, $\frac{T_C}{T_F} << \frac{T}{T_C} << 1$. We show that only the two-quasiparticle scattering process is important in the collision integrals and the form of the Boltzmann equation is the same as normal state. By using the method of Sykes and Brooker [8] we obtain the

Keywords: Quantum liquids; Fermi liquids

viscosity coefficients A,B,C,D and E of the A-phase. The plan of the paper is as follows: in section 2, the collision integral for the Bogoliubov quasiparticles is obtained, then in section 3 the Boltzmann equation is written and the viscosity coefficients are calculated. Section 4 is allocated to the discussion and concluding remarks.

Collision Integral

In a normal Fermi liquid the total quasiparticle number is conserved, and therefore the only scattering processes allowed are those in which the number of quasiparticles in the final state is the same as the number in the initial state. At low temperatures, the density of the excitations is low, and consequently the most important processes are those in which two quasiparticles scatter. The collision integral for a binary collision, specified by $P_1 + P_2 \rightleftarrows P_3 + P_4$, for a normal Fermi liquid is [10]:

$$I(P_1, \uparrow) = -\int d\tau_2\, d\tau_3\, dP_4\, W\, [n_1 n_2 (1-n_3)(1-n_4) - n_3 n_4 (1-n_1)(1-n_2)]\, \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\, \delta(P_1 + P_2 - P_3 - P_4) \quad (1)$$

where i stands for P_i and $d\tau_{pi}$ is the density of states in a momentum interval d_{pi}, n_{pi} is the quasiparticle distribution function and $W = \frac{1}{4} W\uparrow\uparrow + \frac{1}{2} W\uparrow\downarrow$ is the transition probability.

In a superfluid state, the quasiparticle number is not conserved, and other processes as well as the two-quasiparticle scattering process can occur. For example, one quasiparticle can decay into three, or three quasiparticles can coalesce to produce one. To see this in more detail, one has to consider the Bogoliubov transformation between the normal quasiparticle creation and annihilation operators $\alpha^+_{p\sigma}$ and $\alpha_{p\sigma}$ and the creation and annihilation operators $\alpha^+_{p\sigma}$ and $\alpha_{p\sigma}$ in the superfluid state, i.e.

$$a_{p\sigma} = \sum_{\beta} U^{p}_{\sigma\beta}\, \alpha_{p\beta} - V^{p}_{\alpha\beta}\, \alpha^{+}_{-p\beta}$$
$$a^{+}_{p\sigma} = \sum_{\beta} U^{p*}_{\sigma\beta}\, \alpha^{+}_{p\beta} - V^{p*}_{\sigma\beta}\, \alpha_{-p\beta} \quad (2)$$

where the matrix elements $U^{p}_{\sigma\beta}$ and $V^{p}_{\sigma\beta}$ can be chosen for the ABM state as

$$U^{p}_{\alpha\beta} = \left\{\frac{1}{2}\left(1 + \frac{\varepsilon_p}{E_p}\right)\right\}^{1/2} \delta_{\alpha\beta}$$
$$V^{p}_{\alpha\beta} = \left\{\frac{1}{2}\left(1 - \frac{\varepsilon_p}{E_p}\right)\right\}^{1/2} \delta_{\alpha\beta} \quad (3)$$

where $E_p = [\varepsilon_p^2 + |\Delta_p|^2]^{1/2}$ and the spin independent part of Δ_p, the equilibrium gap parameter of the ABM state is given by [11]

$$\Delta_P = \Delta(T) \left[\frac{8\pi}{3}\right]^{1/2} Y_{11}(\hat{P}) \quad (4)$$

where $\Delta(T)$ is the maximum of the gap. The interaction between the quasiparticles in superfluid is found by performing a Bogoliubov transformation on the normal state interaction and is

$$H = \frac{1}{4} \sum_{1,2,3,4} \langle 3,4|t|1,2\rangle\, a^{+}_4 a^{+}_3 a_1 a_2 \quad (5)$$

By substituting equation (2) in equation (5) we get

$$H = \frac{1}{4} \sum_{1,2,3,4} \langle 3,4|t|1,2\rangle (U^{*}_4 \alpha^{+}_4 - V^{*}_4 \alpha_{-4})(U^{*}_3 \alpha^{+}_3 - V^{*}_3 \alpha_{-3}) (U_1 \alpha_1 - V_1 \alpha^{+}_{-1}) \times (U_2 \alpha_2 - V_2 \alpha^{+}_{-2}) \quad (6)$$

where i stands for p_i, σ_i. Note that, H in equation (6) contains terms $\alpha^{+}_4 \alpha^{+}_3 \alpha^{+}_2 \alpha_1$, $\alpha^{+}_4 \alpha_1 \alpha_{-3} \alpha_2$, $\alpha^{+}_4 \alpha^{+}_3 \alpha_2 \alpha_1$, $\alpha^{+}_4 \alpha^{+}_3 \alpha^{+}_2 \alpha^{+}_1$, and $\alpha_{-4} \alpha_{-3} \alpha_2 \alpha_1$ which convert one quasiparticle into three, three quasiparticles into one, two quasiparticles into two and the last two terms create four quasiparticles from the condansate and scatter four quasiparticles into condansate respectively. The last two processes are not allowed, since in each process the energy should be conserved in the collision integral. Therefore, the collision integral in terms of the Bogoliubov quasiparticles may be written as [1,3]

$$I = I_{13} + I_{22} + I_{31} \quad (7)$$

where

$$I_{13} = \frac{(m^*)^3}{4\pi^4 K_B T} \int \frac{d\Omega}{4\pi} \frac{d\phi_2}{2\pi}\, d\varepsilon_2\, d\varepsilon_3\, d\varepsilon_4\, W_{13}\, n^0_1 (1-n^0_2)(1-n^0_3)(1-n^0_4) [\psi_1 - \psi_2 - \psi_3 - \psi_4]\, \delta(E_1 - E_2 - E_3 - E_4), \quad (8)$$

$$I_{22} = \frac{(m^*)^3}{4\pi^4 K_B T} \int \frac{d\Omega}{4\pi} \frac{d\phi_2}{2\pi}\, d\varepsilon_2\, d\varepsilon_3\, d\varepsilon_4\, W_{22}\, n^0_1 n^0_2 (1-n^0_3)(1-n^0_4)$$

$$[\psi_1+\psi_2-\psi_3-\psi_4]\ \delta(E_1+E_2-E_3-E_4), \quad (9)$$

and

$$I_{31}=\frac{(m^*)^3}{4\pi^4 K_B T}\int \frac{d\Omega}{4\pi}\frac{d\phi_2}{2\pi}\, d\varepsilon_2\, d\varepsilon_3\, d\varepsilon_4 W_{31}\, n_1^0 n_2^0 n_3^0 (1-n_4^0)$$
$$[\psi_1+\psi_2+\psi_3-\psi_4]\ \delta(E_1+E_2+E_3-E_4), \quad (10)$$

with the notation of reference [10].* The transition probabilities W_{22}, W_{13} and W_{31} can be written in terms of the matrix elements $U^p_{\alpha\beta}$, $V^p_{\alpha\beta}$ and corresponding normal state collision terms. By substituting equation (3) in the expressions for the transition probability and taking $q=P_4-P_2=P_2-P_1\simeq 0$ for the ABM state at low temperatures (see below), after a bit of algebra we have

$$W_{22}(\uparrow\downarrow)=\frac{|V_0|^2}{4}\left(1-\frac{|\Delta_3||\Delta_2|-\varepsilon_3\varepsilon_2}{E_3E_2}\right)\left(1-\frac{|\Delta_1||\Delta_4|-\varepsilon_1\varepsilon_4}{E_1E_4}\right)$$

$$W_{22}(\uparrow\uparrow)=\frac{|V_0|^2}{4}\left(1-\frac{|\Delta_4||\Delta_3|-\varepsilon_3\varepsilon_4}{E_4E_3}\right)\left(1-\frac{|\Delta_2||\Delta_1|-\varepsilon_2\varepsilon_1}{E_2E_1}\right)$$

$$W_{13}(\uparrow\downarrow)=\frac{|V_0|^2}{4}\left\{1-\frac{|\Delta_3||\Delta_2|+\varepsilon_3\varepsilon_2}{E_3E_2}\right\}\left\{1-\frac{|\Delta_4||\Delta_1|-\varepsilon_4\varepsilon_1}{E_4E_1}\right\}$$

$$W_{13}(\uparrow\uparrow)=\frac{|V_0|^2}{4}\left\{1-\frac{|\Delta_4||\Delta_3|+\varepsilon_4\varepsilon_3}{E_4E_3}\right\}\left\{1-\frac{|\Delta_2||\Delta_1|-\varepsilon_2\varepsilon_1}{E_2E_1}\right\}$$

$$W_{31}(\uparrow\downarrow)=\frac{|V_0|^2}{4}\left\{1-\frac{|\Delta_2||\Delta_1|+\varepsilon_2\varepsilon_1}{E_2E_1}\right\}\left\{1-\frac{|\Delta_4||\Delta_3|-\varepsilon_4\varepsilon_3}{E_4E_3}\right\}$$

$$W_{31}(\uparrow\uparrow)=\frac{|V_0|^2}{4}\left\{1-\frac{|\Delta_4||\Delta_1|-\varepsilon_4\varepsilon_1}{E_4E_1}\right\}\left\{1-\frac{|\Delta_3||\Delta_2|+\varepsilon_3\varepsilon_2}{E_3E_2}\right\} \quad (11)$$

In the low temperatures, $\theta_{p_1}\simeq 0$ and $\varepsilon_{p_1}\simeq 0$ contribute most to the integrals of the viscosity coefficients (see section 3), where θ_{p_1} is the angle between the gap axis, $\hat{l}$, and the momentum of quasiparticle, P_1. The energy delta-function in equation (8) indicates that $\theta_{p_i}\simeq 0$ and $\varepsilon_{p_i}\simeq 0$ (i= 2,3,4) in I_{13}. Consideration of the following formulae

$$\cos\theta_{ij}=\cos\theta_i\cos\theta_j+\sin\theta_i\sin\theta_j\cos(\phi_i-\phi_j),$$
$$\cos\theta_{12}=\cos\theta_{34}=\cos\theta,$$
$$\cos\theta_{13}=\cos\theta_{24}=\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}\cos\phi,$$

*Throughout the paper we take η=1.

and

$$\cos\theta_{14}=\cos\theta_{23}=\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\cos\phi,$$

and the presence of the Fermi function n_2^0 and $n_2^0 n_3^0$ in I_{22} and I_{31} indicates also that $\theta_{p_i}\simeq 0$ (i=2,3,4) in these collision integrals. Substituting the transition probabilities, equation (11), in equations (8)-(10) and using the above results, we see that practically only the $W_{22}(\uparrow\uparrow)$ of the transition probabilities, equation (11), has non-zero value. Hence only the two- quasiparticle scattering process contributes to the collision integrals in the ABM state at low temperatures. Furthermore, by taking $\frac{\varepsilon_{p_i}}{E_{p_j}}\simeq 1$ in the collision integral I_{22} we see that the collision integral for ABM state is:

$$I=I_{22}=\frac{m^*}{8\pi^4\beta}\int\frac{d\Omega}{4\pi}\frac{W_{22}}{\cos\frac{\theta}{2}}\left[\int_\delta^\infty+\int_{-\infty}^{-\delta}dx\right]\left[\int_\delta^\infty+\int_{-\infty}^{-\delta}dy\right]$$
$$n_0(t)\,n_0(x+y-t)$$
$$\{1-n_0(y)\}\ \{1-n_0(x)\}\ [\psi_1+\psi_2-\psi_3-\psi_4] \quad (12)$$

where $x=\beta E_{p_3}$, $y=\beta E_{p_4}$, $t=\beta E_{p_1}$, $\delta=\beta\Delta(0)\theta$ and $\beta=(K_BT)^{-1}$. Integrating with respect to x and y, after a certain amount of simplification and ignoring the terms proportional to δ, we see that the collision integral has the form as normal state [8]. Of course, as we said, $\theta_{p_i}\simeq 0$ contributes most to the integrands of I_{22} which implies $\theta<<1$, and the limits of integration on θ is therefore from 0 to θ_m (see section 4 for the value of θ_m).

Viscosity Coefficients

In the hydrodynamics limit, the distribution function in the streaming terms in the Boltzmann equation may be replaced by local equilibrium distribution function. To calculate the viscosity, one needs to consider a local equilibrium distribution function corresponding to a spatially varying velocity U, $n_p=(e^{\beta}(E_{p-p}.U)+1)^{-1}$. Keeping only terms of first order in the departure from equilibrium, the Boltzmann equation reduces to

$$-\frac{1}{2}n_p'\left[p_i\frac{\partial E_p}{\partial p_k}-\frac{1}{3}p_l\frac{\partial E_p}{\partial p_l}\delta_{ik}\right]\left[\frac{\partial U_i}{\partial r_k}+\frac{\partial U_k}{\partial r_i}-\frac{2}{3}\frac{\partial U_n}{\partial r_n}\delta_{ik}\right]$$
$$=I_{22}(\delta n_p') \quad (13)$$

where $n_p'=\frac{\partial n_p^0}{\partial E_p}$, $\delta n_p'=\delta n_p-n_p'\delta$, E_p characterizes the deviation from local equilibrium and in obtaining the lhs of equation (13) we supposed $U \simeq V^s$ in order to cancel the term $\frac{\partial n_p}{\partial t}$ with the term $-\frac{1}{3}\frac{\partial n_p}{\partial E} p_l \frac{\partial E}{\partial p_l} \nabla . U$.

The fourth rank viscosity tensor may be defined as

$$\pi_{ln} = -\eta_{lnik}\left[\frac{\partial U_i}{\partial r_k} + \frac{\partial U_k}{\partial r_i} - \frac{2}{3}\frac{\partial U_n}{\partial r_n}\delta_{ik}\right] \tag{14}$$

where π_{ln}, the momentum current density, may be expressed in terms of the Bogoliubov quasiparticle distribution function:

$$\pi_{ln} = \sum_p \left[p_l \frac{\partial E_p}{\partial p_n} - \frac{1}{3} p_m \frac{\partial E_p}{\partial p_m}\delta_{ln}\right]\delta n'_p \tag{15}$$

Following Abrikosov and Khalatnikov [10], $\delta n'_p$ can be written in terms of the function $q(E_p, T)$

$$\delta n'_p = \frac{1}{2} n'_p \left[p_i \frac{\partial E_p}{\partial p_k} - \frac{1}{3} p_l \frac{\partial E_p}{\partial p_l}\delta_{ik}\right]\left[\frac{\partial U_i}{\partial r_k} + \frac{\partial U_k}{\partial r_i} - \frac{2}{3}\frac{\partial U_n}{\partial r_n}\delta_{ik}\right]$$

$$q(E_p, T) = n'_p \, \psi_p \tag{16}$$

Comparison of equation (14) with equations (15) and (16) gives

$$\eta_{lnik} = -\frac{3}{2}\frac{m^*}{m}\rho v_F^2 \int \frac{d\Omega_p}{4\pi} d\varepsilon_p \, n'_p \left[\frac{\varepsilon_p}{E_p}\right]^2 g_{ln} g_{ik} \, q(E_p, T)$$

where $g_{ik} = \hat{P}_i \hat{P}_k - \frac{1}{3}\delta_{ik}$. (17)

The viscosity tensor for a system with uniaxial symmetry can be written in terms of the components of symmetry axis, $\hat{l}$, with five coefficients A,B,C,D and E

$$\eta_{lnik} = A(\delta_{li}\delta_{nk} + \delta_{lk}\delta_{ni}) + B(\delta_{li}\hat{l}_n\hat{l}_k + i \leftrightarrow k) + C\delta_{ln}\delta_{ik} + D\hat{l}_l\hat{l}_n\hat{l}_i\hat{l}_k + E(\delta_{ln}\hat{l}_i\hat{l}_k + \delta_{ik}\hat{l}_l\hat{l}_n) \tag{18}$$

Furthermore, the definition of viscosity in equations (14), (15) and (17) indicates that η_{lnik} is a traceless tensor, i.e. $\delta_{ln}\eta_{lnik} = 0$, which gives the following Combescot's equalities [7] between the viscosity coefficients:

$$2A + 3C + E = 0$$
$$4B + D + 3E = 0$$

By taking the gap axis, $\hat{l}$, along the z-axis, we can write the components of the viscosity tensor in terms of the coefficients A,B,C,D and E, then by using equation (17) we have

$$A = \frac{3}{64}\rho v_F^2 \frac{m^*}{m}\beta \int_0^\pi d\theta \sin\theta (1-\cos^2\theta)^2 \int_0^\infty d\varepsilon_p \left[\frac{\varepsilon_p}{E_p}\right]^2 \mathrm{sech}^2(\beta E_p/2)\, q(E_p, T) \tag{19}$$

$$B = \frac{3}{64}\rho v_F^2 \frac{m^*}{m}\beta \int_0^\pi d\theta \sin\theta (-5\cos^4 + 6\cos^2\theta - 1) \int_0^\infty d\varepsilon_p \left[\frac{\varepsilon_p}{E_p}\right]^2 \mathrm{sech}^2(\beta E_p/2)\, q(E_p, T) \tag{20}$$

$$C = \frac{3}{64}\rho v_F^2 \frac{m^*}{m}\beta \int_0^\pi d\theta \sin\theta (\cos^4\theta + \frac{2}{3}\cos^2\theta - \frac{7}{9}) \int_0^\infty d\varepsilon_p \left[\frac{\varepsilon_p}{E_p}\right]^2 \mathrm{sech}^2(\beta E_p/2)\, q(E_p, T) \tag{21}$$

$$D = \frac{3}{64}\rho v_F^2 \frac{m^*}{m}\beta \int_0^\pi d\theta \sin\theta (35\cos^4 - 30\cos^2\theta + 3) \int_0^\infty d\varepsilon_p \left[\frac{\varepsilon_p}{E_p}\right]^2 \mathrm{sech}^2(\beta E_p/2)\, q(E_p, T) \tag{22}$$

$$E = \frac{3}{69}\rho v_F \frac{m^*}{m}\beta \int_0^\pi d\theta \sin\theta (-5\cos^4\theta + 2\cos^2\theta + \frac{1}{3}) \int_0^\infty d\varepsilon_p \left[\frac{\varepsilon_p}{E_p}\right]^2 \mathrm{sech}^2(\beta E_p/2)\, q(E_p, T) \tag{23}$$

At low temperatures, $\varepsilon_p \simeq K_B T$ and the function $\mathrm{sech}^2(\beta E_p/2)$ is practically only non-zero for the values of $\theta_p \simeq 0$ in equations (19)-(23). In the last paragraph of section 2, we used this fact and proved that the collision integral for ABM state has the same form as in the normal state. Hence, $q(E_p, T)$ can be obtained from the Boltzmann equation (13) which is similar to the normal Fermi liquid (of course we keep E_p instead of ε_p and remember that the limits of the integration on θ are 0 and θ_m).

The Boltzmann equation has been solved exactly by Sykes and Brooker [8] and Jensen *et al.* [9]. We use the solution of sykes and Brooker for the kinetic equation. Equation (13) can be written as

$$\frac{\pi^2 + t^2}{2} q(t) - \lambda_2 \int_{-\infty}^{\infty} dx K(x,t)\, q(x) = 0 \tag{24}$$

where

$$K(x,t) = \frac{e^{-t}+1}{e^{-x}+1}\,\frac{(x-t)}{e^{(x-t)}-1} \quad \text{and}$$

$$\lambda_2 = 1 - \frac{3}{4} < W(\theta,\phi)\,(1-\cos\theta)^2 \sin^2\phi >_{\theta_m} / < W(\theta,\phi) >_{\theta_m} \qquad (25)$$

$$\text{where } < A >_{\theta_m} \equiv \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\theta_m} d\theta \sin\theta \frac{A(\theta,\phi)}{\cos\frac{\theta}{2}}$$

By defining the functions $Q(t) = q(t)\,\text{sech}\,\frac{t}{2}$ and

$$\phi(k) = \int_{-\infty}^{\infty} dt e^{ikt} Q(t) \qquad (26)$$

Sykes and Brooker [8] obtained

$$\phi(k) = \frac{\pi\tau_{nm}^{(0,T)}}{4} \sum_{n=0}^{\infty} \frac{(4n+3)}{(n+1)(2n+1)\{(n+1)(2n+1)-\lambda_2\}} p^{1}_{2n+1} (\tanh \pi k) \qquad (27)$$

where

$$\tau_{nm}^{(0,T)} = \frac{32\eta\varepsilon_F \beta^2}{\pi^2 < W(\theta,\phi) >_{\theta_m}} \qquad (28)$$

By using equations (26) - (27) in equations (19) - (23), and doing the integration on t, we get:

$$A = \frac{3}{32}\rho v_F^2 \frac{m^*}{m} \tau_{nm}(0,T)(\Delta(0)\beta)^{-6} \sum_{n=0}^{\infty} \frac{(4n+3)}{(n+1)(2n+1)\{(n+1)(2n+1)-\lambda_2\}} \int_0^{\infty} dy\, y^5 \left\{1 - p_{2n+1}\left[\tanh\frac{y}{2}\right]\right\} \qquad (29)$$

$$B = \frac{3}{8}\rho v_F^2 \frac{m^*}{m} \tau_{nm}(0,T)(\Delta(0)\beta)^{-4} \sum_{n=0}^{\infty} \frac{(4n+3)}{(n+1)(2n+1)\{(n+1)(2n+1)-\lambda_2\}} \int_0^{\infty} dy\, y^3 \left\{1 - p_{2n+1}\left[\tanh\frac{y}{2}\right]\right\} \qquad (30)$$

$$9C = D = -3E = \frac{3}{4}\rho v_F^2 \frac{m^*}{m} \tau_{nm}(0,T)(\Delta(0)\beta)^{-2} \sum_{n=0}^{\infty} \frac{(4n+3)}{(n+1)(2n+1)\{(n+1)(2n+1)-\lambda_2\}} \int_0^{\infty} dy\, y \left\{1 - p_{2n+1}\left[\tanh\frac{y}{2}\right]\right\} \qquad (31)$$

where $p_{2n+1}(x)$ is the Legendre polynomial functions.

Discussions and Concluding Remarks

In this paper, we have calculated the viscosity coefficients of the infinitely extended superfluid ^{3}He-A at low temperatures. Our results can serve as a useful starting point for variational calculations in the intermediate temperature range where our results are not applicable. We can also compare the exact results with the approximate collision integrals [6,7,12], and guess their validity in a whole range of temperatures.

At low temperatures, the mean free path of quasiparticle is larger than characteristic dimensions in the experiments, and therefore, the experimental measurements of the viscosity coefficients would not be the bulk ones. The boundary effects of the apparatus must be taken into account [13,14], but that is beyond the scope of this paper. We may calculate the mean free path, $l(T)$, of Bogoliubov quasiparticle by taking the root mean square of the average distance travelled by a quasiparticle of momentum **P**, which is $L_p = \tau_p \nabla_p E_p = \tau P V_p \frac{\varepsilon_p}{E_p}$. This is given by

$$l(T) = \left[\frac{\sum_p L_p^2 n_p^0}{\sum_p n_p^0}\right]^{1/2} = v_F \left[\int \frac{d\Omega_p}{4\pi} \int_0^{\infty} d\varepsilon_p \tau_p^2 \left[\frac{\varepsilon_p}{E_p}\right]^2 n_p^0 \Big/ \int \frac{d\Omega_p}{4\pi}\int_0^{\infty} d\varepsilon_p n_p^0\right]^{1/2} \qquad (32)$$

Because of the presence of the Fermi function, n_p, in the integrands, only the values of $\theta_p \approx 0$ give non-zero value to the integrands of equation (32) at low temperatures. Following the same procedure as that in section 2, it is easy to show that τ_p can be obtained in a similar way as the normal state quasiparticle lifetime [16]. Hence we write for this limit

$$\tau_p^{-1} = \frac{1}{16\varepsilon_F} < W(\theta,\psi) >_{\theta_m} \frac{(\pi K_B T)^2 + E_p^2}{1+\exp(-E_p/K_B T)} \qquad (33)$$

As we said previously, θ_m is the maximum value of θ at low temperatures. By comparison of the terms E_p^2 and $(\pi K_B T)^2$ in equation (33) we may write $E_p \sim \pi K_B T$. Now θ_m can be found by the fact that n_p^0 has only non-zero value in equation (32) if $\Delta(0)\theta_m = E_p \sim \pi K_B T$, which gives $\theta_m \sim \pi K_B T/\Delta(0)$. The scattering amplitude $W(\theta, \phi)$ is

$$w(\theta,\phi) = \frac{\pi}{4}[3A_t^2 + A_s^2] \qquad (34)$$

where the dimensionless singlet and triplet components A_s and A_t are [5]

$$A_s = S_0 + S_1\cos\theta + S_2\left[\frac{1}{2}(3\cos^2\theta - 1) + \frac{3}{4}(\cos\theta - 1)^2(\cos^2\phi - 1)\right] + \ldots$$

$$A_t = \{T_0 + T_1 \cos\theta + T_2 \frac{1}{2}(3\cos^2\theta - 1) = \dots\} \cos\phi \quad (35)$$

where S_l and T_l are related to the Landau parameters. For small values of θ, the triplet component A_t is zero. This is derived from the fact that the forward scattering sum role indicates $\Sigma_l T_l = 0$.

By substituting equations (34) and (33) in equation (32) and doing the integrations we have

$$l(T) \simeq .35\,\gamma \left[\frac{\Delta(0)}{K_B T_C}\right]^2 l_n \left[\frac{T_C}{T}\right]^4 \quad (36)$$

where $l_n = v_F \tau_n(0, T_C)$ and $\gamma \equiv \langle W(\theta,\phi)\rangle / A_s^2$. The temperature dependence of the mean free path of the quasiparticles, T^{-4}, has also been estimated phenomenologically by Greaves *et al.* [17].

The viscosity coefficients depend strongly on the values of the λ_2 parameters [eq. (25)] which involves the scattering amplitudes and therefore depends on the values of the Landau parameters. Wölfle *et al.* [4] in their calculations of the shear viscosity of B-phase used λ_2 as a parameter which can be determined from other experiments. By using the shear viscosity or the zero-sound attenuation in the normal state, they obtained the values of λ_2 ranging from 0.65 at melting pressure to about 0.7 at 20 bar. The values of λ_2 have also been calculated theoretically by many authors and have been tabulated by Einzel [18] in Table III of his paper.

In this paper, we need the value of λ_2 at low temperatures. By substituting equations (34) and (35) in equation (25) and doing the integration we have

$$\lambda_2 = 1 - \frac{1}{2}\left[\frac{\pi K_B T}{2\Delta(0)}\right]^4 \quad (37)$$

The expression for the viscosity coefficients A,B,C,D and E in equations (29) - (31) contain the series whose first term (n=0) gives the factor $\frac{1}{1-\lambda_2} = 2\left[\frac{2\Delta(0)}{\pi K_B T}\right]^4$. It is clear that the contribution of the other terms in the series is negligible and the viscosity coefficients are

$$A = \frac{1395\gamma}{8\pi}\eta(T_C)(T_C/T)^2 \quad (38)$$

$$B = \frac{336}{\pi^3}\gamma\eta(T_C)\left[\frac{\Delta(0)}{K_B T_C}\right](T_C/T)^4 \quad (39)$$

$$9C = D = -3E = \frac{960}{\pi^5}\gamma\eta(T_C)\left[\frac{\Delta(0)}{K_B T_C}\right]^4 (T_C/T)^6 \quad (40)$$

where $\eta(T_C) = \frac{1}{5}\rho v_F \frac{m^*}{m}\tau_n(0, T_C)$.

By using a simple relaxation time approximation, Combescot [7] obtained the shear viscosity A and B directly. The other coefficients C, D and E were obtained with the help of the first sound attenuation formula. The viscosity coefficients are proportional to the relaxation time $\tau(T)$, which, by using the collision integral and the scaling argument, Combescot, Vollhardt and Wölfle [19, 20] indicated that $\tau(T) \alpha T^{-4}$ at low temperatures, $\frac{T_F}{T_C} \ll \frac{T}{T_C} \ll 1$. As a result, the absolute temperature dependence of the viscosity coefficients are: $A \propto T^2$, B= const, C, D and $E \propto T^{-2}$. Valls *et al.* [12], by defining a second rank viscosity tensor and using the same scaling approach, obtained $\eta_{xy} \propto T^2$, η_{xz} = const and $\eta_{22} \propto T^{-2}$ which are consistent with Combescot's results. Comparison between these results and ours shows the important role of the λ_2 parameter in determining the temperature dependence of the viscosity coefficients, since the appearance of the factor $\frac{1}{1-\lambda_2}$ in the formulae for the viscosity coefficients rises the temperature dependence of these coefficients by T^{-4}.

A more detailed comparison between our exact calculated viscosity coefficients and approximate collision integral approach [6] is worthwhile, since one can see the validity of the approximate approach in the intermediate temperatures. If we modify this approach by saying that λ_2 is not a constant parameter at low temperatures, say $\frac{T_C}{T_F} \ll \frac{T}{T_C} \ll 1$, and should therefore be determined by equation (25), it is easy to show that the viscosity coefficients A,B,C,D and E (eq. (14) of ref. [6]) have the same temperature dependence as equations (38)-(40). The numerical factors are also nearly equal to each other. Bearing in mind the fact that agreement between the results of the exact calculations near T_c^1 and the approximate approach [4] are good, we can say that the latter approach is very useful for intermediate temperatures.

At extremely low temperatures, say $\frac{T}{T_C} \le \frac{T_C}{T_F}$, Combescot [19] showed that the temperature dependence of $\tau(T)$, the quasiparticle relaxation time, is T^{-5}. By using his scaling argument in eq. (13) we may write: $A \propto T^{-3}$, $B \propto T^{-5}$ and $9C = D = -3E \propto T^{-7}$.

To summarize, the viscosity coefficients A,B,C,D and E are calculated at low temperatures. The λ_2 parameter plays an important role in deriving these coefficients. The approximate collision integral may give good results for the bulk viscosity coefficients at intermediate temperatures, since its results at temperatures near T_c and low temperatures are nearly the same as exact results.

References

1. Bhattacharyya, P., Pethick, C.J. and Smith, H. *Phys. Rev.*, **15B**, 3367, (1977).
2. Pethick, C.J., Smith, H. and Bhattacharyya, P. *Ibid.*, 3384, (1977).
3. Shahzamanian, M.A. Some problems in superfluid ^{3}He. Unpublished Ph.D. thesis. University of Sussex, (1975).
4. Wölfle, P. and Einzel, D. *J. Low Temp. Phys.*, **32**, 39, (1978).
5. Ono, Y.A., Hara, J. and Nagai, K. *Ibid.*, **48**, 167, (1982).
6. Shahzamanian, M.A. *Proc. 18th Int. Conf. Low Temp. Phys, Japanese J. Applied Phys.*, **26**, 151, (1987).
7. Combescot, R. *Phys. Rev.*, **12B**, 4839, (1975).
8. Sykes, J. and Brooker, G.A. *Ann. Phys.*, (NY) **56**, 1, (1970).
9. Jensen, H.H., Smith, H. and Wilkins, J.W., *Phys. Lett.*, **27A**, 532, (1968).
10. Abrikosov, A.A. and Khalatnikov, I. M. *Rept. Progr. Phys.*, **22**, 329, (1959).
11. Leggett, A. J. *Rev. Mod. Phys.*, **47**, 415, (1975).
12. Valls, O. and Houghton, A. *Phys. Lett.*, **54A**, 143, (1975).
13. Shahzamanian, M.A. *J. Phys. C.*, **21**, 553, (1988).
14. Einzel, D., Wölfle, P., Jensen, H.H. and Smith, H. *Phys. Rev. Lett*, **52**, 1705, (1984).
15. Einzel, D. and Wölfle, P. *J. Low. Temp. Phys.*, **32**, 19, (1978).
16. Morel, P. and Noziers, P. *Phys. Rev.*, **126**, 1909, (1962).
17. Greaves, N.A. and Leggett, A. J. *J. Phys. C.*, **16**, 4383, (1983).
18. Einzel, D. *J. Low Temp. Phys.*, **54**, 427, (1984).
19. Combescot, R. *Phys. Rev. Lett.*, **35**, 471, (1975).
20. Vollhardt, D. and Wölfle, P. *The superfluid phases of Helium 3*. Taylor and Francis, London, (1990).