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IDEAL J^*-ALGEBRAS

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Abstract

A C^*-algebra A is called an ideal C^*-algebra (or equally a dual algebra) if it is an ideal in its bidual A^{**}. M.C.F. Berglund proved that subalgebras and quotients of ideal C^*-algebras are also ideal C^*-algebras, that a commutative C^*-algebra A is an ideal C^*-algebra if and only if it is isomorphic to $C_0(\Omega)$ for some discrete space Ω. We investigate ideal J^*-algebras and show that the above results can be generalized to that of J^*-algebras. Furthermore, it is proved that if A is an ideal J^*-algebra, then $sp(a^*a)$ has no nonzero limit point for each a in A and consequently A has semifinite rank and is a restricted product of its simple ideals.

Introduction

A J^*-algebra is a closed complex subspace A of the space of all bounded linear transformations from one Hilbert space to another such that $aa^*a \in A$ whenever $a \in A$. J^*-algebras were introduced by Harris in [9], [10], where it was shown that the open unit ball of J^*-algebras are bounded symmetric homogeneous domains and that many holomorphic properties of these domains can be expressed in terms of the algebraic properties of the associated J^*-algebras. Harris also established an algebraic theory for J^*-algebras in analogy to that of C^*-algebras [11].

A C^*-algebra A is called an ideal C^*-algebra if it is an ideal in its bidual A^{**}. Ideal C^*-algebras were defined and studied by F.C.M. Berglund [4], who proved that subalgebras and quotients of ideal C^*-algebras are also ideal C^*-algebras, that a commutative C^*-algebra A is an ideal C^*-algebra if and only if it is isomorphic to the space $C_0(\Omega)$ of all continuous complex-valued functions vanishing at infinity on a discrete space Ω. The object of this paper is to investigate ideal J^*-algebras and show that the above results can be generalized to that of J^*-algebras. It is also shown that if A is an ideal J^*-algebra, then $sp(a^*a)$ has no nonzero limit point for each a in A and consequently A is a restricted product of its simple ideals. For similar results in JB^*-triples see the recent paper by Bunce and Chu [2].

Keywords: Ideal; J^*-algebras; Bidual; Semifinite rank; Ideal J^*-algebras

Definitions and Preliminaries

Suppose H and K are complex Hilbert spaces. Let $B(H, K)$ denote the Banach space of all bounded transformations from H to K with the operator norm. For each element a in $B(H, K)$ there is a uniquely determined element a^* in $B(H, K)$ such that

$$(ax, y) = (x, a^* y) \text{ for all } x \in H \text{ and } y \in K.$$

One calls a^* the adjoint of a.

A closed subspace A of $B(H, K)$ is called a J^*-algebra if $aa^*a \in A$, whenever $a \in A$. Examples of J^*-algebras are C^*-algebras, JC^*-algebras and ternary rings of operators [14]. Furthermore, by the Gelfand-Naimark theorem we may regard B^*-algebras and C^*-ternary ring [14] as examples of J^*-algebras.

Suppose A and B are J^*-algebras. A map $\Phi : A \to B$ is called a J^*-isomorphism if Φ is a bounded linear bijection of A onto B satisfying

$$\Phi(aa^*a) = \Phi(a)\Phi(a)^*\Phi(a).$$

for all $a \in A$. Note that if Φ is a J^*-isomorphism, then Φ is an isometry and conversely every surjective linear isometry Φ of A onto B is a J^*-isomorphism [9, Theorem 4].

Let the map $\Phi : B(H, K) \rightarrow B(H \oplus K)$ be defined by

$$\Phi(a) = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}, a \in B(H, K).$$

Then Φ is a linear isometry of $B(H, K)$ into $B(H \oplus K)$ satisfying,

$$\Phi(ab^* c) = \Phi(a)\Phi(b)^*\Phi(c),$$

for all a, b, c in $B(H, K)$. Hence, every J^*-algebra may be considered as a J^*-subalgebra of some C^*-algebra. This fact was used in the following theorem to prove that the bidual of a J^*-algebra is also a J^*-algebra.

Theorem 2.1. Let A be a J^*-algebra. Then there is a J^*-isomorphism Φ of A onto a J^*-algebra $\Phi(A)$ with the property that (identifying $\Phi(A)$ with its canonical image in $\Phi(A)^{**}$) the identity map on $\Phi(A)$ extends to an isometry of $\Phi(A)^{**}$ onto the weak closure B of $\Phi(A)$. This isometry is a homeomorphism in the w^*-topology of $\Phi(A)^{**}$ and the weak operator topology of B.

Since the weak closure of a J^*-algebra is also a J^*-algebra, it follows from the above theorem that we can regard $A^{**} \cong \Phi(A)^{**}$ as a J^*-algebra which contains A as a J^*-subalgebra.

Suppose A is a J^*-algebra, then by [11, Proposition 1], $ab^* c + cb^* a \in A$, whenever a, b, c are in A. A J^*-ideal in A is a closed subspace J of A such that if a, b, $c \in A$, then $ab^* c + cb^* a \in J$ whenever $b \in J$ or $c \in J$. A J^*-algebra A is simple if the only J^*-ideals in A are $\{0\}$ and A. For example, the set of all compact transformations is a J^*-ideal in $B(H, K)$.

Ideal J^*-Algebras

A J^*-algebra A is said to be an ideal J^*-algebra if it is a J^*-ideal in the bidual A^{**} of A. For example, $A = C(H, K)$, the set of all compact operators is a J^*-ideal in $A^{**} = B(H, K)$ and consequently A is an ideal J^*-algebra. A C^*-algebra is an ideal if and only if it is isomorphic to a C^*-algebra of compact operators [4, Theorem 5.5]. First, we show the hereditary properties of ideal J^*-algebras.

Theorem 3.1. Suppose A is an ideal J^*-algebra. (i) Each J^*-subalgebra of A is an ideal J^*-algebra. (ii) Each quotient of A by its closed J^*-ideal is an ideal J^*-algebra.

Proof. (i) Suppose B is a J^*-subalgebra of A. Regarding $B \subset B^{**}$, $B^{**} \subset A^{**}$ and note that if x is an element of A which is not in B, then there is a functional f in A^* with $f(x) = 1$ and $f|_B = 0$. So $B^{**} \cap A = B$ because B is w^*-dense in B^{**}. Take a, b, c in B^{**}, since A is an ideal J^*-algebra, then $ab^*c + cb^*a \in A$ whenever $b \in B$ or $c \in B$. But B^{**} is a J^*- algebra and so $ab^*c + cb^*a \in B^{**} \cap A = B$, and therefore B is a J^*-ideal in B^{**}. Hence B is an ideal J^*-algebra. (ii) Suppose J is a closed J^*-ideal in A, then by [6, Corollary 5] the quotient space A/J is a J^*-algebra. Now (ii) can be easily proved by using the identification $(A/J)^{**} \cong A^{**}/J^{\perp\perp}$, and the fact that weak closure of a J^*-ideal is also a J^*-ideal [11].

Remark. A closed subspace J of a Banach space X is called an L-summand, if there is a closed subspace J' of X such that $X = J \oplus J'$, and if $x \in J$, $y \in J'$, then

$$\|x + y\| = \|x\| + \|y\|.$$

A subspace J is an M-ideal in X if $J^{\perp}$, the annihilator of J, is an L-summand in X^*. M-ideals are introduced by Alfsen and Effros in [1]. If A is a J^*-algebra, by [3, Theorem 3.2], then the M-ideals in A are exactly the closed J^*-ideals of A. Therefore, a J^*-algebra A is an ideal J^*-algebra if and only if A is an M-ideal in its bidual A^{**}. Banach spaces which are M-ideals in their biduals are introduced and studied by Harmand and Lima in [8]. Hence, Theorem 3.1 can also be proved by applying Theorem 3.4 of [8].

If A is an ideal C^*-algebra, it is proved in [4, Theorem 5.5] that $sp(x)$ has no nonzero limit point for each $x = x^* \in A$. In the case of ideal J^*-algebra, we have the following result.

Theorem 3.2. Suppose A is an ideal J^*-algebra. Then $sp(a^* a)$ has no nonzero limit point for each a in A.

Proof. Suppose a is a nonzero element of A. Let B be the J^*-subalgebra of A generated by a. Then by [12, Proposition 1.2.1.] B is J^*-isomorphic to $C_0(\Omega)$ for some locally compact Hausdorff space Ω.

Suppose $\Phi : B \longrightarrow C_0(\Omega)$ is a J^*-isomorphism. Then the bitranspose $\Phi^{**} : B^{**} \longrightarrow C_0(\Omega)^{**}$ is a surjective isometry and consequently is a J^*-isomorphism. Since A is an ideal J^*-algebra, it follows from Theorem 3.1. (i) that B and therefore $C_0(\Omega)$ are ideal J^*-algebras. By [11, Lemma 3.5] and the above remark, $sp(a^* a)$ has no nonzero limit point.

A J^*-algebra A is said to have semifinite rank if $sp(a^* a)$ has no nonzero limit point for each $a \in A$ [11]. The next result follows from the above theorem and [11, Theorems 3.3 and 5.9].

Corollary 3.3. (i) Each ideal J^*-algebra has semifinite rank. (ii) Each ideal J^*-algebra is generated by its minimal isometries. (iii) Each ideal J^*-algebra is J^*-isomorphism to a restricted product of its simple ideals. (iv) For each closed J^*-ideal J of an ideal J^*-algebra A there is a closed J^*-ideal I such that $A = I \oplus J$.

Commutative Ideal J^*-Algebras

Suppose Ω is a locally compact Hausdorff space and $A = C_0(\Omega)$ is an ideal J^*-algebra, then Ω must be discrete. To prove it, suppose E is a compact subset of Ω and $J = \{f \in A; f|_E = 0\}$. Then $C_0(\Omega)/J \cong C(E)$, and Theorem 3.1. (i), $C(E)$ is an ideal in $C(E)^{**}$. However, E is compact and so $C(E)$ has an identity. So $C(E)$ must be reflexive and consequently E is discrete. A J^*-algebra of semifinite rank is called commutative in [13], if each minimal partial isometry of A is central. Since each ideal J^*-algebra has semifinite rank, we have the following characterization of commutative ideal J^*-algebras.

Theorem 4.1. A commutative J^*-algebra A is an ideal J^*-algebra if and only if it is J^*-isomorphic to $C_0(\Omega)$, where Ω is discrete.

Proof. Apply Theorem 2.2 of [13].

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