Vol.10 No.4
Autumn 1999

THE EXISTENCE AND UNIQUENESS OF THE SOLUTION OF THE SPECTRAL PROBLEM II

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Abstract

Following our previous project [1], we are going to prove the existence and uniqueness of the solution of the spectral problem in this project. First, we have proven the uniqueness of the solution. Then to prove the existence, we construct the adjoint problem corresponding to this spectral problem. Next, the uniqueness of the adjoint problem will be proven. Finally, we use the fact that the uniqueness of the adjoint problem is the existence of the main problem as discussed by [2] and [3]. We have determined the existence of the spectral problem. The paper will conclude with three unsolved problems.

Introduction

In [1] the mixed problem, (i.e. initial and boundary value problem BVP), including the time dependent Schrödinger equation with non-local boundary condition, is investigated. In this investigation, we have obtained a spectral problem as a BVP with non-local boundary condtions. By applying the Laplace transform EXT, sufficient conditions are defined such that the spectral problem can be reduced to second Fredholm integral equations.

Buzurnguk and Serov [4] have studied the problem at a defined amplitude $q(x) = v\left(\frac{1}{|x-x_o|}\right)\frac{1}{|x-x_o|}\beta$, $x \in \mathbb{R}^2$ and $\mathbb{R}^3$, for the Schrödinger operator with a singular potential. Birman and Laptev [5] have considered $H(\alpha)=-\Delta-\alpha\, v(x)$, where α is a real parameter, $\alpha>0$, $x \in \mathbb{R}^d$, d=2, for the negative discrete spectrum of a two dimensional Schrödinger operator. The corresponding eigenfunction properties have been studied, few spectra and Weyl-type asymptotic functions were given at $d \geq 3$. Grinberg [6] studied the spectral and scattering properties of the three-dimensional anisotropic Schrödinger operator. The operator was in the form of $H=-\Delta+v$ with a small zero-order pseudo-differential potential. An eigenfunction expansion was given. The existence and completeness of the wave operators were proven.

In [1] and [7], the solutions of the Schrödinger equation with initial and boundary conditions at two and three dimensions on the half band and half cylinder space have been considered. In these studies, the problem which will be referred to as mixed problem, has been analysed by applying the Laplace transform. In [7], a mixed problem was considered and two interesting unsolved problems were proposed. In [1], one boundary value problem, including Schrödinger's equation with non-local and general conditions, was considered. The transformed problem, in

Key words: Spectral problem; Mixed problem; Non-local boundary condition; Time dependent Schrödinger equation; Adjoint problem

some conditions will be in the form of the second type; Fredholm's integral equations.

The Problem Proposition

In this article, the existence and uniquencess of the solution of the following boundary problem will be examined:

$$\Delta\tilde{u}(x,\lambda) + \frac{2\mu}{h^2}(ih\lambda - v(x))\,\tilde{u}(x,\lambda) = \frac{2\mu i}{h}\psi(x), \qquad x\in D \qquad (1)$$

$$\sum_{k=1}^{2}\left[\sum_{i=1}^{2}\alpha_{pj}^{(k)}(x_1)\frac{\partial\tilde{u}(x,\lambda)}{\partial x_j} + \alpha_{p0}^{(k)}(x_1)\,\tilde{u}(x,\lambda)\right]\Big|_{x_2=\gamma_k(x_1)} = \tilde{\alpha}_p(x_1,\lambda) \qquad (2)$$

$$x_1\in[a_1,b_1]; \quad p=1,2$$

where $i=\sqrt{-1}$ and μ and h are real, positive physical constants. $v(x)$ and $\psi(x)$, $x\in D$ are known, real and differentiable functions, and *D* is a domain of the plane surface. $\tilde{u}(x,\lambda)$, $x\in D$ and $\lambda\in ¢$ (¢ is a complex plane) is an unknown function with complex values. The boundary conditions (2) for p=1,2 gives two relations which are linear and independent of each other, the coefficients on the left hand side are the known functions. $\tilde{\alpha}_p(x_1,\lambda)$ are complex and continuous functions. Γ is a boundary of convex domain $D\subset\mathbb{R}^2$, ($\mathbb{R}^2$ is a real plane), divided for two curves Γ_1, Γ_2 by two lines at $x_1=a_1$ and $x_1=b_1$ parallel to x_2, ($x_2\perp x_1$) then for $x_1\in(a_1,b_1)$ functions $\gamma_1(x_1)$ and $\gamma_2(x_1)$ will be related for Γ_1 and Γ_2 respectively, $\gamma_1(x_1)<\gamma_2(x_1)$ but, $\gamma_1(a_1)=\gamma_2(a_1)$ and $\gamma_1(b_1)=\gamma_2(b_1)$, [1].

Construction of the Adjoint Problem

Considering the homogeneous form of Equation (1) as:

$$\ell\tilde{u} \equiv \Delta\tilde{u}(x,\lambda) + \frac{2\mu}{h^2}(ih\lambda - v(x))\,\tilde{u}(x,\lambda) \qquad (1_1)$$

and multiplying that by $\overline{\tilde{v}(x,\lambda)}$ and integrating on domain D as follows:

$$\int_D \Delta\tilde{u}(x,\lambda).\overline{\tilde{v}(x,\lambda)}\,dx + \frac{2\mu}{h^2}\int_D(ih\lambda - v(x))\,\tilde{u}(x,\lambda)\overline{\tilde{v}(x,\lambda)}\,dx =$$

$$\int_\Gamma\sum_{j=1}^{2}\left[\frac{\partial\tilde{u}(x,\lambda)}{\partial x_j}\overline{\tilde{v}(x,\lambda)} - \tilde{u}(x,\lambda)\overline{\frac{\partial\tilde{v}(x,\lambda)}{\partial x_j}}\right]\cos(\nu,x_j)\,dx +$$

$$\int_D\tilde{u}(x,\lambda)\left[\overline{\Delta\tilde{v}(x,\lambda)} + \frac{2\mu}{h^2}(ih\lambda - v(x))\,\overline{\tilde{v}(x,\lambda)}\right]dx \qquad (3)$$

Then, the adjoint operator of (1_1) can be obtained as follows:

$$\ell^{*}\tilde{v} \equiv \Delta\tilde{v}(x,\lambda) - \frac{2\mu}{h^2}(ih\rho + v(x))\tilde{v}(x,\lambda) \qquad (4)$$

where $\rho=\bar{\lambda}$ is the conjugate parameter, ν is a vector perpendicular to the Γ boundary directed ouside of domain D. The boundary conditions have determined that the first term on the right hand side of the Rel. (3) would be zero.

$$\int_\Gamma\sum_{j=1}^{2}\left[\frac{\partial\tilde{u}(x,\lambda)}{\partial x_j}\overline{\tilde{v}(x,\lambda)} - \tilde{u}(x,\lambda)\overline{\frac{\partial\tilde{v}(x,\lambda)}{\partial x_j}}\right]\cos(\nu,x_j)\,dx = 0$$

The integrand Γ is changed to Γ_j, (j=1,2) and then is changed to (a_1, b_1) as shown below:

$$\sum_{k=1}^{2}\sum_{j=1}^{2}\int_{a_1}^{b_1}\left[\frac{\partial\tilde{u}(x,\lambda)}{\partial x_j}\overline{\tilde{v}(x,\lambda)} - \tilde{u}(x,\lambda)\overline{\frac{\partial\tilde{v}(x,\lambda)}{\partial x_j}}\right]\Big|_{x_2=\gamma_k(x_1)}\cos(\nu_k,x_j)\frac{dx_1}{\cos(x_1,\tau_k)} =$$

$$\int_{a_1}^{b_1}\left[\tilde{u}'(x_1,\gamma_1(x_1),\lambda)\,\overline{\tilde{v}(x_1,\gamma_1(x_1),\lambda)}\,\gamma'_1(x_1) - \tilde{u}'(x_1,\gamma_2(x_1),\lambda)\overline{\tilde{v}(x_1,\gamma_2(x_1),\lambda)}\,\gamma'_2(x_1)\right]dx_1$$

$$-\int_{a_1}^{b_1}\left[\frac{\partial\tilde{u}(x,\lambda)}{\partial x_2}\Big|_{x_2=\gamma_1(x_1)}\left(1+\gamma'^2_1(x_1)\right)\overline{\tilde{v}(x_1,\lambda_1(x_1),\lambda)} - \frac{\partial\tilde{u}(x,\lambda)}{\partial x_2}\right]\Big|_{x_2=\gamma_2(x_1)}\left(1+\gamma'^2_2(x_1)\right)$$

$$\overline{\tilde{v}(x_1,\lambda_1(x_1),\lambda)}\,]\,dx_1 + \int_{a_1}^{b_1}\tilde{u}(x_1,\gamma_1(x_1),\lambda)\left[\overline{\frac{\partial\tilde{v}(x,\lambda)}{\partial x_2}}\Big|_{x_2=\gamma_1(x_1)} - \overline{\frac{\partial\tilde{v}(x,\lambda)}{\partial x_1}}\Big|_{x_2=\gamma_1(x_1)}\gamma_1(x_1)\right]dx_1 -$$

$$\int_{a_1}^{b_1}\tilde{u}(x_1,\gamma_1(x_1),\lambda)\left[\overline{\frac{\partial\tilde{v}(x,\lambda)}{\partial x_2}}\Big|_{x_2=\gamma_2(x_1)} - \overline{\frac{\partial\tilde{v}(x,\lambda)}{\partial x_1}}\Big|_{x_2=\gamma_2(x_1)}\gamma_1(x_1)\right]dx_1 = 0 \qquad (5)$$

where τ_k is a vectror tangent to Γ_k directed to a moving point direction on the boundary. To define the function $\frac{\partial\tilde{u}(x,\lambda)}{\partial x_2}\Big|_{x_2=\gamma_k(x_1)}$, (k=1,2) the homogeneous equation of (2) is taken as:

$$\sum_{k=1}^{2}\alpha_{p1}^{(k)}(x_1)\frac{\partial\tilde{u}(x,\lambda)}{\partial x_1}\Big|_{x_2=\gamma_k(x_1)} + \sum_{k=1}^{2}\alpha_{p2}^{(k)}\frac{\partial\tilde{u}(x,\lambda)}{\partial x_2}\Big|_{x_2=\gamma_k(x_1)} + \sum_{k=1}^{2}\alpha_{p0}^{(k)}(x_1)\tilde{u}(x_1,\gamma_k(x_1),\lambda) =$$

$$\sum_{k=1}^{2}\alpha_{p1}^{(k)}(x_1)\,u'(x_1,\gamma_k(x_1),\lambda) + \sum_{k=1}^{2}\left(\alpha_{p1}^{(k)}(x_1) - \alpha_{p1}^{(k)}(x_1)\gamma'_k(x_1)\right)\frac{\partial\tilde{u}(x,\lambda)}{\partial x_2}\Big|_{x_2=\gamma_k(x_1)} +$$

$$\sum_{k=1}^{2}\alpha_{p0}^{(k)}(x_1)\,\tilde{u}(x_1,\gamma_k(x_1),\lambda) = 0 \qquad P=1,2$$

under the condition of:

$$\Delta(x_1) = \begin{vmatrix}\alpha_{12}^{(1)}(x_1) - \alpha_{11}^{(1)}(x_1)\gamma'_1(x_1) & \alpha_{12}^{(2)}(x_1) - \alpha_{11}^{(2)}(x_1)\gamma'_2(x_1)\\ \alpha_{22}^{(1)}(x_1) - \alpha_{21}^{(1)}(x_1)\gamma'_1(x_1) & \alpha_{22}^{(2)}(x_1) - \alpha_{21}^{(2)}(x_1)\gamma'_2(x_1)\end{vmatrix} \neq 0 \qquad (6)$$

then, by applying Cramer's rule:

$$\frac{\partial\tilde{u}(x,\lambda)}{\partial x_2}\Big|_{x_2=\gamma_1(x_1)} = \frac{-1}{\Delta(x_1)}\begin{vmatrix}\sum_{k=1}^{2}\left[\alpha_{11}^{(k)}(x_1)\tilde{u}'(x_1,\gamma_k(x_1),\lambda) + \alpha_{10}^{(k)}(x_1)\tilde{u}(x_1,\gamma_k(x_1),\lambda)\right] & \alpha_{12}^{(2)}(x_1) - \alpha_{11}^{(2)}(x_1)\gamma'_2(x_1)\\ \sum_{k=1}^{2}\left[\alpha_{21}^{(k)}(x_1)\tilde{u}'(x_1,\gamma_k(x_1),\lambda) + \alpha_{20}^{(k)}(x_1)\tilde{u}(x_1,\gamma_k(x_1),\lambda)\right] & \alpha_{22}^{(2)}(x_1) - \alpha_{21}^{(2)}(x_1)\gamma'_2(x_1)\end{vmatrix}$$

$$\frac{\partial\tilde{u}(x,\lambda)}{\partial x_2}\Big|_{x_2=\gamma_2(x_1)} = \frac{-1}{\Delta(x_1)}\begin{vmatrix}\alpha_{12}^{(1)}(x_1) - \alpha_{11}^{(1)}(x_1)\gamma'_1(x_1) & \sum_{k=1}^{2}\left[\alpha_{11}^{(k)}(x_1)\tilde{u}'(x_1,\gamma_k(x_1),\lambda) + \alpha_{10}^{(k)}(x_1)\tilde{u}(x_1,\gamma_k(x_1),\lambda)\right]\\ \alpha_{22}^{(1)}(x_1) - \alpha_{21}^{(1)}(x_1)\gamma'_1(x_1) & \sum_{k=1}^{2}\left[\alpha_{21}^{(k)}(x_1)\tilde{u}'(x_1,\gamma_k(x_1),\lambda) + \alpha_{20}^{(k)}(x_1)\tilde{u}(x_1,\gamma_k(x_1),\lambda)\right]\end{vmatrix}$$

The above relation is substituted in Rel. 5 and the terms are rewritten as follows:

$$\int_{a_1}^{b_1} \tilde{u}(x_1,\gamma_1(x_1),\lambda)\overline{\tilde{v}(x_1,\gamma_1(x_1),\lambda)}\left[\gamma_1(x_1)+\frac{1+\gamma_1^2(x_1)}{\Delta(x_1)}\begin{vmatrix}\alpha_{11}^{(1)}(x_1) & \alpha_{12}^{(2)}(x_1)-\alpha_{11}^{(2)}(x_1)\gamma'_2(x_1)\\ \alpha_{21}^{(1)}(x_1) & \alpha_{22}^{(2)}(x_1)-\alpha_{21}^{(2)}(x_1)\gamma'_2(x_1)\end{vmatrix}\right]dx_1 -$$

$$-\int_{a_1}^{b_1} \tilde{u}(x_1,\gamma_2(x_1),\lambda)\overline{\tilde{v}(x_1,\gamma_2(x_1),\lambda)}\left[\gamma_2(x_1)+\frac{1+\gamma_2^2(x_1)}{\Delta(x_1)}\begin{vmatrix}\alpha_{12}^{(1)}(x_1)-\alpha_{11}^{(1)}(x_1)\gamma'_1(x_1) & \alpha_{11}^{(2)}(x_1)\\ \alpha_{22}^{(1)}(x_1)-\alpha_{21}^{(1)}(x_1)\gamma_1(x_1) & \alpha_{21}^{(2)}(x_1)\end{vmatrix}\right]dx_1 +$$

$$+\int_{a_1}^{b_1} \frac{1+\gamma_1^2(x_1)}{\Delta(x_1)}\tilde{u}(x_1,\gamma_2(x_1),\lambda), \overline{\tilde{v}(x_1,\gamma_1(x_1),\lambda)}\begin{vmatrix}\alpha_{11}^{(2)}(x_1) & \alpha_{12}^{(2)}(x_1)-\alpha_{11}^{(2)}(x_1)\gamma'_2(x_1)\\ \alpha_{21}^{(2)}(x_1) & \alpha_{22}^{(2)}(x_1)-\alpha_{21}^{(2)}(x_1)\gamma'_2(x_1)\end{vmatrix}dx_1 -$$

$$-\int_{a_1}^{b_1} \frac{1+\gamma_2^2(x_1)}{\Delta(x_1)}\tilde{u}(x_1,\gamma_1(x_1),\lambda), \overline{\tilde{v}(x_1,\gamma_2(x_1),\lambda)}\begin{vmatrix}\alpha_{12}^{(2)}(x_1)-\alpha_{11}^{(1)}(x_1)\gamma'_1(x_1) & \alpha_{11}^{(1)}(x_1)\\ \alpha_{22}^{(1)}(x_1)-\alpha_{21}^{(1)}(x_1)\gamma'_1(x_1) & \alpha_{21}^{(1)}(x_1)\end{vmatrix}dx_1 +$$

$$+\int_{a_1}^{b_1} \tilde{u}(x_1,\gamma_1(x_1),\lambda)\Big[\overline{\frac{\partial\tilde{v}(x,\lambda)}{\partial x_2}}\Big|_{x_2=\gamma_1(x_1)} - \overline{\frac{\partial\tilde{v}(x,\lambda)}{\partial x_1}}\Big|_{x_2=\gamma_1(x_1)} \gamma_1(x_1)+\frac{1+\gamma_1^2(x_1)}{\Delta(x_1)}\begin{vmatrix}\alpha_{10}^{(1)}(x_1) & \alpha_{12}^{(2)}(x_1)-\alpha_{11}^{(2)}(x_1)\gamma'_2(x_1)\\ \alpha_{20}^{(1)}(x_1) & \alpha_{22}^{(2)}(x_1)-\alpha_{21}^{(2)}(x_1)\gamma'_2(x_1)\end{vmatrix}$$

$$\overline{\tilde{v}(x_1,\gamma_1(x_1),\lambda)} - \frac{1+\gamma_2^2(x_1)}{\Delta(x_1)}\begin{vmatrix}\alpha_{12}^{(1)}(x_1)-\alpha_{11}^{(1)}(x_1)\gamma'_1(x_1) & \alpha_{10}^{(1)}(x_1)\\ \alpha_{22}^{(1)}(x_1)-\alpha_{21}^{(1)}(x_1)\gamma'_1(x_1) & \alpha_{20}^{(1)}(x_1)\end{vmatrix}\overline{\tilde{v}(x,\gamma_2(x_1),\lambda)}\Big]dx_1 -$$

$$-\int_{a_1}^{b_1} \tilde{u}(x_1,\gamma_2(x_1),\lambda)\Big[\overline{\frac{\partial\tilde{v}(x,\lambda)}{\partial x_2}}\Big|_{x_2=\gamma_2(x_1)} - \overline{\frac{\partial\tilde{v}(x,\lambda)}{\partial x_1}}\Big|_{x_2=\gamma_2(x_1)} \gamma'_2(x_1)+\frac{1+\gamma_1^2(x_1)}{\Delta(x_1)}\begin{vmatrix}\alpha_{10}^{(2)}(x_1) & \alpha_{12}^{(2)}(x_1)-\alpha_{11}^{(2)}(x_1)\gamma_2(x_1)\\ \alpha_{20}^{(2)}(x_1) & \alpha_{22}^{(2)}(x_1)-\alpha_{21}^{(2)}(x_1)\gamma_2(x_1)\end{vmatrix}$$

$$\overline{\tilde{v}(x_1,\gamma_1(x_1),\lambda)} + \frac{1+\gamma_2^2(x_1)}{\Delta(x_1)}\begin{vmatrix}\alpha_{12}^{(1)}(x_1)-\alpha_{11}^{(1)}(x_1)\gamma_1(x_1) & \alpha_{10}^{(2)}(x_1)\\ \alpha_{22}^{(1)}(x_1)-\alpha_{21}^{(1)}(x_1)\gamma_1(x_1) & \alpha_{20}^{(2)}(x_1)\end{vmatrix}\overline{\tilde{v}(x_1,\gamma_2(x_1),\lambda)}\Big]dx_1 = 0 \qquad (7)$$

If, in the above relation, the inside of the bracket of the first term is shown by $A_1(x_1)$,

$$A_1(x_1) \equiv \gamma'_1(x_1) + \frac{1+\gamma'^2_1(x_1)}{\Delta(x_1)}\begin{vmatrix}\alpha_{11}^{(1)}(x_1) & \alpha_{12}^{(2)}(x_1)-\alpha_{11}^{(2)}(x_1)\gamma_2(x_1)\\ \alpha_{21}^{(1)}(x_1) & \alpha_{22}^{(2)}(x_1)-\alpha_{21}^{(2)}(x_1)\gamma_2(x_1)\end{vmatrix} =$$

$$= \frac{1}{\Delta(x_1)}\Big\{\big[\alpha_{11}^{(2)}(x_1)\alpha_{22}^{(1)}(x_1)-\alpha_{12}^{(1)}(x_1)\alpha_{21}^{(2)}(x_1)\big]\gamma_1(x_1)\gamma_2(x_1)+\big[\alpha_{12}^{(1)}(x_1)\alpha_{22}^{(2)}(x_1)-\alpha_{12}^{(2)}(x_1)\alpha_{22}^{(1)}(x_1)\big]\gamma_1(x_1)$$

$$-\big[\alpha_{11}^{(1)}(x_1)\alpha_{21}^{(2)}(x_1)-\alpha_{11}^{(2)}(x_1)\alpha_{21}^{(1)}(x_1)\big]\gamma_2(x_1)+\big[\alpha_{11}^{(1)}(x_1)\alpha_{22}^{(2)}(x_1)-\alpha_{12}^{(2)}(x_1)\alpha_{21}^{(1)}(x_1)\big]\Big\} \qquad (8)$$

and, under the condition of:

$$\Delta_o(x_1) = \begin{vmatrix}\alpha_{11}^{(1)}(x_1) & \alpha_{11}^{(2)}(x_1)\\ \alpha_{21}^{(1)}(x_1) & \alpha_{21}^{(2)}(x_1)\end{vmatrix} \neq 0 \qquad (6_1)$$

the out-come will be:

$$\lim_{x_1\to a_1} A_1(x_1) = \frac{1}{\Delta_o(a_1)}\left[\alpha_{11}^{(2)}(a_1)\alpha_{22}^{(1)}(a_1) - \alpha_{12}^{(1)}(a_1)\alpha_{21}^{(2)}(a_1)\right],$$

$$\lim_{x_1\to b_1} A_1(x_1) = \frac{1}{\Delta_o(b_1)}\left[\alpha_{11}^{(2)}(b_1)\alpha_{22}^{(1)}(b_1) - \alpha_{12}^{(1)}(b_1)\alpha_{21}^{(2)}(b_1)\right] \qquad (9)$$

Similar operations are carried out on the second term of Rel. (7) to obtain $A_2(x_1)$ and relations correspond to Equations (8), (6_1) and (9). The third and fourth terms of Rel. (7) will give $A_3(x_1)$ and $A_4(x_1)$ as follows:

$$A_3(x_1) = \frac{1+\gamma_1^2(x_1)}{\Delta(x_1)}\begin{vmatrix}\alpha_{11}^{(2)}(x_1) & \alpha_{12}^{(2)}(x_1)\\ \alpha_{21}^{(2)}(x_1) & \alpha_{22}^{(2)}(x_1)\end{vmatrix}$$

$$A_4(x_1) = -\frac{1+\gamma_1^2(x_1)}{\Delta(x_1)}\begin{vmatrix}\alpha_{12}^{(1)}(x_1) & \alpha_{11}^{(1)}(x_1)\\ \alpha_{22}^{(1)}(x_1) & \alpha_{21}^{(1)}(x_1)\end{vmatrix}$$

At the points $(x_1 = a_1, b_1)$ we have:

$$\sum_{q=1}^{4} A_q(x_1) = 0 \qquad x_1 = a_1, \quad x_1 = b_1 \qquad (10)$$

If, four of the first terms of Rel. (7) are integrated by parts, then the boundary conditions of the adjoint problems will be as follows:

$$\frac{\partial\tilde{v}(x,\lambda)}{\partial x_2}\Big|_{x_2=\gamma_1(x_1)} - \frac{\partial\tilde{v}(x,\lambda)}{\partial x_1}\Big|_{x_2=\gamma_1(x_1)}\gamma_1(x_1) + \frac{1+\gamma_1^2(x_1)}{\Delta(x_1)}\begin{bmatrix}\overline{\alpha_{10}^{(1)}}(x_1) & \overline{\alpha_{12}^{(2)}}(x_1)-\overline{\alpha_{11}^{(2)}}(x_1)\gamma_2(x_1)\\ \overline{\alpha_{20}^{(1)}}(x_1) & \overline{\alpha_{22}^{(2)}}(x_1)-\overline{\alpha_{21}^{(2)}}(x_1)\gamma_2(x_1)\end{bmatrix}\tilde{v}(x_1,\gamma_1(x_1),\lambda)$$

$$-\frac{1+\gamma_1^2(x_1)}{\Delta(x_1)}\begin{vmatrix}\overline{\alpha_{12}^{(1)}}(x_1)-\overline{\alpha_{11}^{(1)}}(x_1)\gamma_1(x_1) & \overline{\alpha_{10}^{(1)}}(x_1)\\ \overline{\alpha_{22}^{(1)}}(x_1)-\overline{\alpha_{21}^{(1)}}(x_1)\gamma_1(x_1) & \overline{\alpha_{20}^{(1)}}(x_1)\end{vmatrix}\tilde{v}(x_1,\gamma_1(x_1),\lambda)-$$

$$-\left[\overline{A_1(x_1)}\tilde{v}(x_1,\gamma_1(x_1),\lambda)-\right]' - \left[\overline{A_4(x_1)}\tilde{v}(x_1,\gamma_2(x_1),\lambda)-\right]' = 0 \qquad (11)$$

$$\frac{\partial\tilde{v}(x,\lambda)}{\partial x_2}\Big|_{x_2=\gamma_2(x_1)} - \frac{\partial\tilde{v}(x,\lambda)}{\partial x_1}\Big|_{x_2=\gamma_2(x_1)}\gamma_2(x_1) - \frac{1+\gamma_1^2(x_1)}{\Delta(x_1)}\begin{vmatrix}\overline{\alpha_{10}^{(2)}}(x_1) & \overline{\alpha_{12}^{(2)}}(x_1)-\overline{\alpha_{11}^{(2)}}(x_1)\gamma_2(x_1)\\ \overline{\alpha_{20}^{(2)}}(x_1) & \overline{\alpha_{22}^{(2)}}(x_1)-\overline{\alpha_{21}^{(2)}}(x_1)\gamma_2(x_1)\end{vmatrix}\tilde{v}(x_1,\gamma_1(x_1),\lambda)+$$

$$\frac{1+\gamma_2^2(x_1)}{\Delta(x_1)}\begin{vmatrix}\overline{\alpha_{12}^{(1)}}(x_1)-\overline{\alpha_{11}^{(1)}}(x_1)\gamma_1(x_1) & \overline{\alpha_{10}^{(2)}}(x_1)\\ \overline{\alpha_{22}^{(1)}}(x_1)-\overline{\alpha_{21}^{(1)}}(x_1)\gamma_1(x_1) & \overline{\alpha_{20}^{(2)}}(x_1)\end{vmatrix}\tilde{v}(x_1,\gamma_2(x_1),\lambda)-$$

$$-\left[\overline{A_2(x_1)}\tilde{v}(x_1,\gamma_2(x_1),\lambda)\right]' - \left[\overline{A_3(x_1)}\tilde{v}(x_1,\gamma_1(x_1),\lambda)\right]' = 0$$

However, the following statement can be proven:

Theorem1. If the conditions (6), (6_1) and (10) are maintained, also $A_q(x_1) \in C'[a_1, b_1]$, $q=\overline{1,4}$ and the coefficients $\alpha_{pj}^{(k)}(x_1)$, $x_1 \in (a_1,b_1)$, (p, k and j = 1,2) are continuous, then the boundary Γ is a Liapunov line. Consequently, the adjoint problem will be in the form of (4), and (11).

The Unique Solution of the Spectral Problem

The homogeneous form of Equation 1 will be multiplied by $\overline{\tilde{u}(x,\lambda)}$ and integrated over domain D and for the first term, the Ostrogzadsky Gauss law is applied, we obtain:

$$\sum_{j=1}^{2}\int_\Gamma \frac{\partial\tilde{u}(x,\lambda)}{\partial x_j}\overline{\tilde{u}(x,\lambda)}\cos(\nu,x_j)\,dx - \int_D |\text{grad}\,\tilde{u}(x,\lambda)|^2\,dx +$$

$$+\frac{2\mu}{h^2}\int_D (ih\lambda - v(x))\,|\tilde{u}(x,\lambda)|^2\,dx = 0$$

As Rel. (5), if the integrand of the first term of above equation is changed,

$$\int_{a_1}^{b_1}\left[\tilde{u}'(x_1,\gamma_1(x_1),\lambda)\overline{\tilde{u}(x_1,\gamma_1(x_1),\lambda)}\gamma'_1(x_1)-\tilde{u}(x_1,\gamma_2(x_1),\lambda)\overline{\tilde{u}(x_1,\gamma_2(x_1),\lambda)}\gamma'_2(x_1)\right]dx_1 -$$

$$-\int_{a_1}^{b_1}\left[\frac{\partial\tilde{u}(x,\lambda)}{\partial x_2}\Big|_{x_2=\gamma_1(x_1)}(1+\gamma_1'^2(x_1))\,\overline{\tilde{u}(x_1,\gamma_1(x_1),\lambda)}-\frac{\partial\tilde{u}(x,\lambda)}{\partial x_2}\Big|_{x_2=\gamma_2(x_1)}(1+\gamma_2'^2(x_1))\,\overline{\tilde{u}(x_1,\gamma_2(x_1),\lambda)}\right]dx_1$$

$$-\int_D|\text{grad}\,\tilde{u}(x,\lambda)|^2dx+\frac{2\mu}{h^2}\int_D(i\lambda h-v(x))|\tilde{u}(x,\lambda)|^2\,dx=0$$

From the homogeneus form of (2), we have otained $\frac{\partial\tilde{u}(x,\lambda)}{\partial x_2}\Big|_{x_2=\gamma_k(x_1)}$, k=1,2 Also, considering the above relation and Rel. (7), we have:

$$\int_{a_1}^{b_1}\tilde{u}(x_1,\gamma_1(x_1),\lambda)\overline{\tilde{u}(x_1,\gamma_1(x_1),\lambda)}\left[\gamma'_1(x_1)+\frac{1+\gamma_1'^2(x_1)}{\Delta(x_1)}\begin{vmatrix}\alpha_{11}^{(1)}(x_1) & \alpha_{12}^{(2)}(x_1)-\alpha_{11}^{(2)}(x_1)\gamma'_2(x_1)\\ \alpha_{21}^{(1)}(x_1) & \alpha_{22}^{(2)}(x_1)-\alpha_{21}^{(2)}(x_1)\gamma'_2(x_1)\end{vmatrix}\right]dx_1 -$$

$$-\int_{a_1}^{b_1}\tilde{u}(x_1,\gamma_2(x_1),\lambda)\overline{\tilde{u}(x_1,\gamma_2(x_1),\lambda)}\left[\gamma'_2(x_1)+\frac{1+\gamma_2'^2(x_1)}{\Delta(x_1)}\begin{vmatrix}\alpha_{12}^{(1)}(x_1)-\alpha_{11}^{(1)}(x_1)\gamma'_1(x_1) & \alpha_{11}^{(2)}(x_1)\\ \alpha_{22}^{(1)}(x_1)-\alpha_{21}^{(1)}(x_1)\gamma'_1(x_1) & \alpha_{21}^{(2)}(x_1)\end{vmatrix}\right]dx_1 +$$

$$+\int_{a_1}^{b_1}\frac{1+\gamma_1'^2(x_1)}{\Delta(x_1)}\tilde{u}(x_1,\gamma_2(x_1),\lambda)\overline{\tilde{u}(x_1,\gamma_1(x_1),\lambda)}\begin{vmatrix}\alpha_{11}^{(2)}(x_1) & \alpha_{12}^{(2)}(x_1)-\alpha_{11}^{(2)}(x_1)\gamma'_2(x_1)\\ \alpha_{21}^{(2)}(x_1) & \alpha_{22}^{(2)}(x_1)-\alpha_{21}^{(2)}(x_1)\gamma'_2(x_1)\end{vmatrix}dx_1 -$$

$$-\int_{a_1}^{b_1}\frac{1+\gamma_2'^2(x_1)}{\Delta(x_1)}\tilde{u}(x_1,\gamma_1(x_1),\lambda)\overline{\tilde{u}(x_1,\gamma_2(x_1),\lambda)}\begin{vmatrix}\alpha_{12}^{(1)}(x_1)-\alpha_{11}^{(1)}(x_1)\gamma'_1(x_1) & \alpha_{11}^{(1)}(x_1)\\ \alpha_{22}^{(1)}(x_1)-\alpha_{21}^{(2)}(x_1)\gamma'_1(x_1) & \alpha_{21}^{(1)}(x_1)\end{vmatrix}dx_1 +$$

$$+\int_{a_1}^{b_1}|\tilde{u}(x_1,\gamma_1(x_1),\lambda)|^2\frac{1+\gamma_1'^2(x_1)}{\Delta(x_1)}\begin{vmatrix}\alpha_{10}^{(1)}(x_1) & \alpha_{12}^{(2)}(x_1)-\alpha_{11}^{(2)}(x_1)\gamma'_2(x_1)\\ \alpha_{20}^{(1)}(x_1) & \alpha_{22}^{(2)}(x_1)-\alpha_{21}^{(2)}(x_1)\gamma'_2(x_1)\end{vmatrix}dx_1 -$$

$$-\int_{a_1}^{b_1}\frac{1+\gamma_2'^2(x_1)}{\Delta(x_1)}\begin{vmatrix}\alpha_{12}^{(1)}(x_1)-\alpha_{11}^{(1)}(x_1)\gamma'_1(x_1) & \alpha_{10}^{(1)}(x_1)\\ \alpha_{22}^{(1)}(x_1)-\alpha_{21}^{(1)}(x_1)\gamma'_1(x_1) & \alpha_{20}^{(1)}(x_1)\end{vmatrix}\tilde{u}(x_1,\gamma_1(x_1),\lambda)\overline{\tilde{u}(x_1,\gamma_2(x_1),\lambda)}dx_1 +$$

$$+\int_{a_1}^{b_1}\frac{1+\gamma_1'^2(x_1)}{\Delta(x_1)}\begin{vmatrix}\alpha_{10}^{(2)}(x_1) & \alpha_{12}^{(2)}(x_1)-\alpha_{11}^{(2)}(x_1)\gamma'_2(x_1)\\ \alpha_{20}^{(2)}(x_1) & \alpha_{22}^{(2)}(x_1)-\alpha_{21}^{(2)}(x_1)\gamma'_1(x_1)\end{vmatrix}\tilde{u}(x_1,\gamma_2(x_1),\lambda)\overline{\tilde{u}(x_1,\gamma_1(x_1),\lambda)}dx_1 -$$

$$-\int_{a_1}^{b_1}\frac{1+\gamma_1'^2(x_1)}{\Delta(x_1)}\begin{vmatrix}\alpha_{12}^{(1)}(x_1)-\alpha_{11}^{(1)}(x_1)\gamma'_1(x_1) & \alpha_{10}^{(2)}(x_1)\\ \alpha_{22}^{(1)}(x_1)-\alpha_{21}^{(1)}(x_1)\gamma'_1(x_1) & \alpha_{20}^{(2)}(x_1)\end{vmatrix}|\tilde{u}(x_1,\gamma_2(x_1),\lambda)|^2\,dx_1 -$$

$$-\int_D|\text{grad}\,\tilde{u}(x,\lambda)|^2\,dx+\frac{2\mu}{h^2}\int_D(i\lambda h-v(x))|\tilde{u}(x,\lambda)|^2dx=0 \quad (12)$$

The first term of Equation (12) is integrated by parts and considering Equation (8), we can obtain:

$$\int_{a_1}^{b_1}\tilde{u}'(x_1,\gamma_1(x_1),\lambda)\overline{\tilde{u}(x_1,\gamma_1(x_1),\lambda)}\,A_1(x_1)\,dx_1=$$

$$\frac{1}{2}\int_{a_1}^{b_1}A_1(x_1)\left[\tilde{u}'(x_1,\gamma_1(x_1),\lambda)\overline{\tilde{u}(x_1,\gamma_1(x_1),\lambda)}-\tilde{u}(x_1,\gamma_1(x_1),\lambda)\overline{\tilde{u}'(x_1,\gamma_1(x_1),\lambda)}\right]dx_1+$$

$$+\frac{1}{2}|\tilde{u}(x_1,\gamma_1(x_1),\lambda)|^2A_1(x_1)\Big|_{a_1}^{b_1}-\frac{1}{2}\int_{a_1}^{b_1}A'_1(x_1)|\tilde{u}(x_1,\gamma_1(x_1),\lambda)|^2\,dx_1 \quad (12_1)$$

Similar operations for the terms of 2, 3, and 4 in (12) will give:

$$\int_{a_1}^{b_1}\tilde{u}'(x_1,\gamma_2(x_1),\lambda)\overline{\tilde{u}(x_1,\gamma_2(x_1),\lambda)}\,A_2(x_1)\,dx_1=$$

$$\frac{1}{2}\int_{a_1}^{b_1}A_2(x_1)\left[\tilde{u}'(x_1,\gamma_2(x_1),\lambda)\overline{\tilde{u}(x_1,\gamma_2(x_1),\lambda)}-\tilde{u}(x_1,\gamma_2(x_1),\lambda)\overline{\tilde{u}'(x_1,\gamma_2(x_1),\lambda)}\right]dx_1$$

$$+\frac{1}{2}|\tilde{u}(x_1,\gamma_2(x_1),\lambda)|^2A_2(x_1)\Big|_{a_1}^{b_1}-\frac{1}{2}\int_{a_1}^{b_1}A'_2(x_1)|\tilde{u}(x_1,\gamma_2(x_1),\lambda)|^2\,dx_1 \quad (12_2)$$

$$\int_{a_1}^{b_1}\tilde{u}'(x_1,\gamma_2(x_1),\lambda)\overline{\tilde{u}(x_1,\gamma_1(x_1),\lambda)}\,A_3(x_1)\,dx_1=$$

$$\frac{1}{2}\int_{a_1}^{b_1}A_3(x_1)\left[\tilde{u}'(x_1,\gamma_2(x_1),\lambda)\overline{\tilde{u}(x_1,\gamma_1(x_1),\lambda)}-\tilde{u}(x_1,\gamma_2(x_1),\lambda)\overline{\tilde{u}'(x_1,\gamma_1(x_1),\lambda)}\right]dx_1+$$

$$\frac{1}{2}\tilde{u}(x_1,\gamma_2(x_1),\lambda)\overline{\tilde{u}(x_1,\gamma_1(x_1),\lambda)}A_3(x_1)\Big|_{a_1}^{b_1}-\frac{1}{2}\int_{a_1}^{b_1}A'_3(x_1)\tilde{u}(x_1,\gamma_2(x_1),\lambda)\overline{\tilde{u}(x_1,\gamma_1(x_1),\lambda)}\,dx_1 \quad (12_3)$$

$$\int_{a_1}^{b_1}\tilde{u}'(x_1,\gamma_1(x_1),\lambda)\overline{\tilde{u}(x_1,\gamma_2(x_1),\lambda)}\,A_4(x_1)\,dx_1=$$

$$\frac{1}{2}\int_{a_1}^{b_1}A_4(x_1)\left[\tilde{u}'(x_1,\gamma_1(x_1),\lambda)\overline{\tilde{u}(x_1,\gamma_2(x_1),\lambda)}-\tilde{u}(x_1,\gamma_1(x_1),\lambda)\overline{\tilde{u}'(x_1,\gamma_2(x_1),\lambda)}\right]dx_1+$$

$$(12_4)$$

$$+\frac{1}{2}\tilde{u}(x_1,\gamma_1(x_1),\lambda)\overline{\tilde{u}(x_1,\gamma_2(x_1),\lambda)}A_4(x_1)\Big|_{a_1}^{b_1}-\frac{1}{2}\int_{a_1}^{b_1}A'_4(x_1)\tilde{u}(x_1,\gamma_1(x_1),\lambda)\overline{\tilde{u}(x_1,\gamma_2(x_1),\lambda)}\,dx_1$$

Considering the relations (12_1) to (12_4) and the conditions (10), then Rel. (12) will be as follows:

$$\frac{1}{2}\int_{a_1}^{b_1}A_1(x_1)\left[\tilde{u}'(x_1,\gamma_1(x_1),\lambda)\overline{\tilde{u}(x_1,\gamma_1(x_1),\lambda)}-\tilde{u}(x_1,\gamma_1(x_1),\lambda)\overline{\tilde{u}'(x_1,\gamma_1(x_1),\lambda)}\right]dx_1-$$

$$-\frac{1}{2}\int_{a_1}^{b_1}A'_1(x_1)|\tilde{u}(x_1,\gamma_1(x_1),\lambda)|^2\,dx_1+$$

$$+\frac{1}{2}\int_{a_1}^{b_1}A_2(x_1)\left[\tilde{u}'(x_1,\gamma_2(x_1),\lambda)\overline{\tilde{u}(x_1,\gamma_2(x_1),\lambda)}-A_4(x_1)\tilde{u}(x_1,\gamma_2(x_1),\lambda)\overline{\tilde{u}'(x_1,\gamma_2(x_1),\lambda)}\right]+$$

$$-\frac{1}{2}\int_{a_1}^{b_1}A'_2(x_1)|\tilde{u}(x_1,\gamma_2(x_1),\lambda)|^2\,dx_1+$$

$$+\frac{1}{2}\int_{a_1}^{b_1}\{\left[A_3(x_1)\overline{u}(x_1,\gamma_2(x_1),\lambda)\overline{\overline{u}(x_1,\gamma_1(x_1),\lambda)}-A_4(x_1)\overline{u}(x_1,\gamma_2(x_1),\lambda)\overline{\overline{u}(x_1,\gamma_1(x_1),\lambda)}\right]+$$

$$+\left[A_4(x_1)\overline{u}(x_1,\gamma_1(x_1),\lambda)\overline{\overline{u}(x_1,\gamma_2(x_1),\lambda)}-A_3(x_1)\overline{u}(x_1,\gamma_1(x_1),\lambda)\overline{\overline{u}(x_1,\gamma_2(x_1),\lambda)}+\right]\}dx_1-$$

$$-\frac{1}{2}\int_{a_1}^{b_1}A'_3(x_1)\overline{u}(x_1,\gamma_2(x_1),\lambda)\overline{\overline{u}(x_1,\gamma_1(x_1),\lambda)}\,dx_1-\frac{1}{2}\int_{a_1}^{b_1}A'_4(x_1)\overline{u}(x_1,\gamma_1(x_1),\lambda)\overline{\overline{u}(x_1,\gamma_2(x_1),\lambda)}\,dx_1+$$

$$+\int_{a_1}^{b_1}|\overline{u}(x_1,\gamma_1(x_1),\lambda)|^2\frac{1+\gamma_1^2(x_1)}{\Delta(x_1)}\begin{vmatrix}\alpha_{10}^{(1)}(x_1) & \alpha_{12}^{(2)}(x_1)-\alpha_{11}^{(2)}(x_1)\gamma_2(x_1)\\ \alpha_{20}^{(1)}(x_1) & \alpha_{22}^{(2)}(x_1)-\alpha_{21}^{(2)}(x_1)\gamma_2(x_1)\end{vmatrix}dx_1-$$

$$-\int_{a_1}^{b_1}\frac{1+\gamma_2^2(x_1)}{\Delta(x_1)}\begin{vmatrix}\alpha_{12}^{(1)}(x_1)-\alpha_{11}^{(1)}(x_1)\gamma_1(x_1) & \alpha_{10}^{(1)}(x_1)\\ \alpha_{22}^{(1)}(x_1)-\alpha_{21}^{(1)}(x_1)\gamma_1(x_1) & \alpha_{20}^{(1)}(x_1)\end{vmatrix}\overline{u}(x_1,\gamma_1(x_1),\lambda)\overline{\overline{u}(x_1,\gamma_2(x_1),\lambda)}\,dx_1+$$

$$+\int_{a_1}^{b_1}\frac{1+\gamma_1^2(x_1)}{\Delta(x_1)}\begin{vmatrix}\alpha_{10}^{(2)}(x_1) & \alpha_{12}^{(2)}(x_1)-\alpha_{11}^{(2)}(x_1)\gamma_2(x_1)\\ \alpha_{20}^{(2)}(x_1) & \alpha_{22}^{(2)}(x_1)-\alpha_{21}^{(2)}(x_1)\gamma_2(x_1)\end{vmatrix}\overline{u}(x_1,\gamma_2(x_1),\lambda)\overline{\overline{u}(x_1,\gamma_1(x_1),\lambda)}\,dx_1-$$

$$-\int_{a_1}^{b_1}\frac{1+\gamma_2^2(x_1)}{\Delta(x_1)}\begin{vmatrix}\alpha_{12}^{(1)}(x_1)-\alpha_{11}^{(1)}(x_1)\gamma_1(x_1) & \alpha_{10}^{(2)}(x_1)\\ \alpha_{22}^{(1)}(x_1)-\alpha_{21}^{(1)}(x_1)\gamma_1(x_1) & \alpha_{20}^{(2)}(x_1)\end{vmatrix}|\overline{u}(x_1,\gamma_2(x_1),\lambda)|^2\,dx_1-$$

$$-\int_D|\text{grad}\,\overline{u}(x,\lambda)|^2dx+\frac{2\mu}{h^2}\int_D(i\lambda h-v(x))|\overline{u}(x,\lambda)|^2\,dx=0 \quad (13)$$

If in Rel.(13), the following conditions, which are called "condition A", is maintained. "condition A"

$\text{Re}A_q(x_1)=0,\ q=(1,2),$ $\quad\text{Im}\,A'_q(x_1)\le 0,\ q=(1,2)$

$\text{Re}[A_3(x_1)+A_4(x_1)=0,$ $\quad\text{Im}\,[A_3(x_1)-A_4(x_1)]=0$

$$\text{Im}\frac{\begin{vmatrix}\alpha_{10}^{(1)}(x_1) & \alpha_{12}^{(2)}(x_1)-\alpha_{11}^{(2)}(x_1)\gamma_2(x_1)\\ \alpha_{20}^{(1)}(x_1) & \alpha_{22}^{(2)}(x_1)-\alpha_{21}^{(2)}(x_1)\gamma_2(x_1)\end{vmatrix}}{\Delta(x_1)}\ge 0,\quad \text{Im}\frac{\begin{vmatrix}\alpha_{12}^{(1)}(x_1)-\alpha_{11}^{(1)}(x_1)\gamma_1(x_1) & \alpha_{10}^{(2)}(x_1)\\ \alpha_{22}^{(1)}(x_1)-\alpha_{21}^{(1)}(x_1)\gamma_1(x_1) & \alpha_{20}^{(2)}(x_1)\end{vmatrix}}{\Delta(x_1)}\le 0$$

$$\text{Im}\left\{\left[\frac{1+\gamma_1^2(x_1)}{\Delta(x_1)}\begin{vmatrix}\alpha_{10}^{(2)}(x_1) & \alpha_{12}^{(2)}(x_1)-\alpha_{11}^{(2)}(x_1)\gamma_2(x_1)\\ \alpha_{20}^{(2)}(x_1) & \alpha_{22}^{(2)}(x_1)-\alpha_{21}^{(2)}(x_1)\gamma_2(x_1)\end{vmatrix}-\frac{1}{2}A'_3(x_1)\right]+\right.$$

$$\left.\left[-\frac{1+\gamma_2^2(x_1)}{\Delta(x_1)}\begin{vmatrix}\alpha_{12}^{(1)}(x_1)-\alpha_{11}^{(1)}(x_1)\gamma_1(x_1) & \alpha_{10}^{(1)}(x_1)\\ \alpha_{22}^{(1)}(x_1)-\alpha_{21}^{(1)}(x_1)\gamma_1(x_1) & \alpha_{20}^{(1)}(x_1)\end{vmatrix}-\frac{1}{2}A'_4(x_1)\right]\right\}=0,$$

$$\text{Re}\left\{\left[\frac{1+\gamma_1^2(x_1)}{\Delta(x_1)}\begin{vmatrix}\alpha_{10}^{(2)}(x_1) & \alpha_{12}^{(2)}(x_1)-\alpha_{11}^{(2)}(x_1)\gamma_2(x_1)\\ \alpha_{20}^{(2)}(x_1) & \alpha_{22}^{(2)}(x_1)-\alpha_{21}^{(2)}(x_1)\gamma_2(x_1)\end{vmatrix}-\frac{1}{2}A'_3(x_1)\right]-\right.$$

$$\left.\left[-\frac{1+\gamma_2^2(x_1)}{\Delta(x_1)}\begin{vmatrix}\alpha_{12}^{(1)}(x_1)-\alpha_{11}^{(1)}(x_1)\gamma_1(x_1) & \alpha_{10}^{(1)}(x_1)\\ \alpha_{22}^{(1)}(x_1)-\alpha_{21}^{(1)}(x_1)\gamma_1(x_1) & \alpha_{20}^{(1)}(x_1)\end{vmatrix}-\frac{1}{2}A'_4(x_1)\right]\right\}=0$$

The following statement will be proven:

Theorm 2. If the conditions of theorem 1 and condition A are met, then for $\lambda_1\ge 0$, the uniqueness of spectral problem will be proven.

Remark. To restore theorem 2, it is required that the coefficients of the imaginary part of Rel. (13) be set to zero.

The Uniqueness of the Solution for the Adjoint Problem

To prove the uniqueness of the adjoint problem, the homogeneous form of Eq. (4) is multiplied by $\widetilde{\overline{v}}(x,\lambda)$ and integrated over D,

$$\sum_{j=1}^{2}\int_\Gamma\frac{\partial\widetilde{v}(x,\lambda)}{\partial x_j}\overline{\widetilde{v}(x,\lambda)}\cos(v,x_j)\,dx-\int_D|\text{grad}\,\widetilde{v}(x,\lambda)|^2dx-$$

$$-\frac{2\mu}{h^2}\int_D\left(ih\overline{\lambda}+v(x)\right)|\widetilde{v}(x,\lambda)|^2dx=0 \quad (14)$$

and in the boundary condition for the adjoint problem (i.e. Equation. (11), the following terms will be chosen as:

$\overline{\Delta(x_1)}-\overline{\Delta(x_1)}\,\overline{A_1(x_1)}\gamma_1(x_1)=\beta_{12}^{(1)}(x_1),\quad -\overline{\Delta(x_1)}\gamma_1(x_1)-\overline{\Delta(x_1)}\,\overline{A_1(x_1)}=\beta_{11}^{(1)}(x_1),$

$-\overline{\Delta(x_1)}\,\overline{A_4(x_1)}=\beta_{11}^{(2)}(x_1),\quad -\overline{\Delta(x_1)}\,\overline{A_4(x_1)}\gamma_2(x_1)=\beta_{12}^{(2)}(x_1),$

$$\left(1+\gamma_1^2(x_1)\right)\begin{vmatrix}\overline{\alpha_{10}^{(1)}(x_1)} & \overline{\alpha_{12}^{(2)}(x_1)-\alpha_{11}^{(2)}(x_1)\gamma_2(x_1)}\\ \overline{\alpha_{20}^{(1)}(x_1)} & \overline{\alpha_{22}^{(2)}(x_1)-\alpha_{21}^{(2)}(x_1)\gamma_2(x_1)}\end{vmatrix}-\overline{\Delta(x_1)\,A'_1(x_1)}=\beta_{10}^{(1)}(x_1),$$

$$-\left(1+\gamma_2^2(x_1)\right)\begin{vmatrix}\overline{\alpha_{12}^{(1)}(x_1)-\alpha_{11}^{(1)}(x_1)\gamma_1(x_1)} & \overline{\alpha_{10}^{(1)}(x_1)}\\ \overline{\alpha_{22}^{(1)}(x_1)-\alpha_{21}^{(1)}(x_1)\gamma_1(x_1)} & \overline{\alpha_{10}^{(1)}(x_1)}\end{vmatrix}-\overline{\Delta(x_1)\,A'_4(x_1)}=\beta_{10}^{(2)}(x_1)\quad (15_1)$$

and

$\overline{\Delta(x_1)}-\overline{\Delta(x_1)}\,\overline{A_2(x_1)}\gamma_2(x_1)=\beta_{22}^{(2)}(x_1),\quad -\overline{\Delta(x_1)}\gamma_2(x_1)-\overline{\Delta(x_1)}\,\overline{A_2(x_1)}=\beta_{21}^{(2)}(x_1),$

$-\overline{\Delta(x_1)}\,\overline{A_3(x_1)}=\beta_{21}^{(1)}(x_1),\quad -\overline{\Delta(x_1)}\,\overline{A_3(x_1)}\gamma_1(x_1)=\beta_{22}^{(1)}(x_1),$

$$-\left(1+\gamma_1^2(x_1)\right)\begin{vmatrix}\overline{\alpha_{10}^{(2)}(x_1)} & \overline{\alpha_{12}^{(2)}(x_1)-\alpha_{11}^{(2)}(x_1)\gamma_2(x_1)}\\ \overline{\alpha_{20}^{(2)}(x_1)} & \overline{\alpha_{22}^{(2)}(x_1)-\alpha_{21}^{(2)}(x_1)\gamma_2(x_1)}\end{vmatrix}-\overline{\Delta(x_1)\,A'_3(x_1)}=\beta_{20}^{(1)}(x_1),$$

$$\left(1+\gamma_2^2(x_1)\right)\begin{vmatrix}\overline{\alpha_{12}^{(1)}(x_1)-\alpha_{11}^{(1)}(x_1)\gamma_1(x_1)} & \overline{\alpha_{10}^{(2)}(x_1)}\\ \overline{\alpha_{22}^{(1)}(x_1)-\alpha_{21}^{(1)}(x_1)\gamma_1(x_1)} & \overline{\alpha_{20}^{(2)}(x_1)}\end{vmatrix}-\overline{\Delta(x_1)\,A'_2(x_1)}=\beta_{20}^{(2)}(x_1)\quad (15_2)$$

Then, the boundary condition will be as:

$$\sum_{k=1}^{2}\left[\sum_{j=1}^{2}\beta_{pj}^{(k)}(x_1)\frac{\partial\widetilde{v}(x,\lambda)}{\partial x_j}+\beta_{po}^{(k)}(x_1)\,\widetilde{v}(x,\lambda)\right]\Bigg|_{x_2=\gamma_k(x_1)}=0\quad \begin{matrix}k=1,2\\ x\in[a_1,b_1]\end{matrix}\quad (15)$$

Using the same approach to obtain a solution for the adjoint problem requires "condition B" which is equivalent to "condition A" as:

"Condition B"

$R_eB_q(x_1)=0,\quad (q=1,2),$ $\quad\text{Im}\,B_q(x_1)\ge 0,\ q=(1,2)$

$\text{Re}\,[B_3(x_1)+B_4(x_1)]=0,$ $\quad\text{Im}\,[B_3(x_1)-B_4(x_1)]=0$

$$\text{Im}\frac{\begin{vmatrix}\beta_{10}^{(1)}(x_1) & \beta_{12}^{(2)}(x_1)-\beta_{11}^{(2)}(x_1)\gamma_2(x_1)\\ \beta_{20}^{(1)}(x_1) & \beta_{22}^{(2)}(x_1)-\beta_{21}^{(2)}(x_1)\gamma_2(x_1)\end{vmatrix}}{\delta(x_1)}\le 0,\quad \text{Im}\frac{\begin{vmatrix}\beta_{12}^{(1)}(x_1)-\beta_{11}^{(1)}(x_1)\gamma_1(x_1) & \beta_{10}^{(2)}(x_1)\\ \beta_{22}^{(1)}(x_1)-\beta_{21}^{(1)}(x_1)\gamma_1(x_1) & \beta_{20}^{(2)}(x_1)\end{vmatrix}}{\delta(x_1)}\ge 0$$

$$\text{Im}\left\{\left[\frac{1+\gamma_1^2(x_1)}{\delta(x_1)}\begin{vmatrix}\beta_{10}^{(2)}(x_1) & \beta_{12}^{(2)}(x_1)-\beta_{11}^{(2)}(x_1)\gamma_2(x_1)\\ \beta_{20}^{(2)}(x_1) & \beta_{22}^{(2)}(x_1)-\beta_{21}^{(2)}(x_1)\gamma_2(x_1)\end{vmatrix}-\frac{1}{2}B'_3(x_1)\right]+\right.$$

$$\left[-\frac{1+\gamma_2^2(x_1)}{\delta(x_1)}\left|\begin{matrix}\beta_{12}^{(1)}(x_1)-\beta_{11}^{(1)}(x_1)\gamma_1(x_1) & \beta_{10}^{(1)}(x_1)\\ \beta_{22}^{(1)}(x_1)-\beta_{21}^{(1)}(x_1)\gamma_1(x_1) & \beta_{20}^{(1)}(x_1)\end{matrix}\right| -\frac{1}{2}B'_4(x_1)\right]\} = 0$$

$$\mathrm{Re}\ \{\left[\frac{1+\gamma_1^2(x_1)}{\delta(x_1)}\left|\begin{matrix}\beta_{10}^{(2)}(x_1) & \beta_{12}^{(2)}(x_1)-\beta_{11}^{(2)}(x_1)\gamma_2(x_1)\\ \beta_{20}^{(2)}(x_1) & \beta_{22}^{(2)}(x_1)-\beta_{21}^{(2)}(x_1)\gamma_2(x_1)\end{matrix}\right| -\frac{1}{2}B'_3(x_1)\right] -$$

$$\left\{\left[-\frac{1+\gamma_2^2(x_1)}{\delta(x_1)}\left|\begin{matrix}\beta_{12}^{(1)}(x_1)-\beta_{11}^{(1)}(x_1)\gamma_1(x_1) & \beta_{10}^{(1)}(x_1)\\ \beta_{22}^{(1)}(x_1)-\beta_{21}^{(1)}(x_1)\gamma_1(x_1) & \beta_{20}^{(1)}(x_1)\end{matrix}\right| -\frac{1}{2}B'_4(x_1)\right]\right\} = 0$$

From boundary conditions (2), we defined $\alpha_{pj}^{(k)}(x_1)$, $p = 1,2$, $j = \overline{0,2}$; $k=1,2$ and $\Delta(x_1)$ $\Delta_o(x_1)$ for $A_q(x_1)$, $q = \overline{1,4}$; by the same procedure for boundary conditions (15), we define their equivalent, as $\beta_{pj}^{(k)}(x_1)$, $p = 1,2$, $j=0,2$; $k=1,2$ and $\delta(x_1), \delta_o(x_1)\, \beta_q(x_1)$, $q = \overline{0,4}$; respectively. To sum everything or all of this up, the following theorem will be concluded:

Theorem 3. If the conditions for theorem 1 and condition B are satisfied, then for $\lambda \geq 0$, the uniqueness for the adjoint problem is proven. By this definition, the crucial theorem 4 can be concluded.

Theorem 4. Assume that, $\mu > 0$, $h > 0$, $v(x) > 0$, and $x \in D$ is continuous $\psi(x)$ and $\widetilde{\alpha}_p(x_1,\lambda)$, $x_1 \in [a_1,b_1]$ are complex and continuous functions and the boundary conditions (2) are linear, independent functions, $D \subset \mathbb{R}^2$ at the direction of axis x_2 is convex and the boundary Γ is a Liapunov line, $\overline{D} = D \cup \Gamma$, $\Gamma = \Gamma_1 \cup \Gamma_2$, $x_2 = \gamma_k(x_1)$, $k = 1,2$; $\gamma_1(x_1) < \gamma_2(x_1)$ and $[a_1,b_1] = \mathrm{proj} \mid x_1 D$. If the conditions (6), (6_1) and (10), also $\delta(x_1) \neq 0$, $\delta_o(x_1) \neq 0$, and $A_q(x_1) \in C^{(1)}[a_1, b_1]$, $B_q(x_1) \in C^{(1)}[a_1, b_1]$, $q = \overline{1,4}$, are satisfied. Finally $\sum_{q=1}^{4} B_q(x_1) = 0$, $x_1 = (a_1, b_1)$ and the main conditions of A and B are maintained, and the rest of the coefficients are continuous, then, for $\lambda 1 \geq 0$, the existence and uniqueness of the solution for the spectral problem is proven.

Unsolved Problems

1) Consider the solution of Equation (1) when the Global integral is added to the boundary condition (2) as in the equation below:

$$\sum_{s=1}^{2}\left[\sum_{j=1}^{2}\alpha_{pj}^{(s)}(x_1)\frac{\partial\tilde{u}(x,\lambda)}{\partial x_j}+\alpha_{po}^{(s)}(x_1)\,\tilde{u}(x,\lambda)\right]\Bigg|_{x_2=\gamma_s(x_1)} +$$

$$\sum_{s=1}^{2}\int_{a_1}^{b_1}\left[\sum_{j=1}^{2}k_{pj}^{(s)}(x_1,\xi_1)\frac{\partial\tilde{u}(\xi,\lambda)}{\partial x_j}+k_{po}^{(s)}(x_1,\xi_1)\,\tilde{u}(\xi,\lambda)\right]\Bigg|_{\xi_2=\gamma_s(\xi_1)} d\xi = \tilde{\alpha}_p(x_1,\lambda)$$

$$p=1,2 \qquad x_1 \in [a_1,b_1]$$

2) Consider the solution for Equation (1), when the boundary problem is as in the problem below:

$$ih\frac{\partial u(x)}{\partial x_o}+\frac{h^2}{2\mu}\frac{\partial^2 u(x)}{\partial x_1^2}-v(x_1)\,u(x)=0, \qquad x=(x_o,x_1)\in D\subset R^2$$

$$\sum_{s=1}^{2}\left[\sum_{j=0}^{1}\alpha_{pj}^{(s)}(x_0)\frac{\partial u(x)}{\partial x_j}+\alpha_{p}^{(s)}(x_0)\,u(x)\right]\Bigg|_{x_1=\gamma_s(x_0)} = \alpha_p(x_0), \quad p=1,2, \quad x_0\in[a_1,b_1]$$

3) In the above problem, (unsolved problem 2), $v(x_1) \equiv 0$, defines three boundary conditions which are linear and independent and the problem would have infinite solutions:

$$ih\frac{\partial u(x)}{\partial x_o}+\frac{h^2}{2\mu}\frac{\partial^2 u(x)}{\partial x_1^2}=0, \qquad x=(x_o,x_1)\in D\subset R^2 \tag{16}$$

In addition, add a non-local boundary condition (i.g.a Global integral) to the (16) so that the problem would have no solution, hint [7].

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