Vol. 11 No. 1
Winter 2000

J. Sci. I.R. Iran

TWO LOW-ORDER METHODS FOR THE NUMERICAL EVALUATION OF CAUCHY PRINCIPAL VALUE INTEGRALS OF OSCILLATORY KIND

G.E. Okecha

Department of Mathematics, University of Namibia, P.O. Box 25344 Windhoek, Namibia

Abstract

In this paper, we develop two piecewise polynomial methods for the numerical evaluation of Cauchy Principal Value integrals of oscillatory kind. The two piecewise-polynomial quadratures are compact, easy to implement, and are numerically stable. Two numerical examples are presented to illustrate the two rules developed. The convergence of the two schemes is proved and some error bounds obtained.

1. Introduction

In this paper, we are concerned with the numerical evaluation of Cauchy principal value integrals of oscillatory kind of the form

$$I(w, s) = \int_a^b \frac{e^{iwy} g(y)}{y-s}\, dy,\ w \geq 0,\ i^2 = -1, \qquad a < s < b \qquad (1)$$

where $g(y)$ is analytic in $a<y<b$ and $g(s) = 0$.

The integral, as noted in [8] has two practical problems. It is oscillatory and has a singularity of Cauchy type. The numerical evaluation of integrals of this type has wide applications in applied mathematics, physics and engineering. For more information on methods of evaluating integrals of this form, you may see, [5, 6, 7, 9, 10] for examples.

We consider two low-order quadrature rules for the numerical evaluation of (1) which are implicitly the modified trapezium and Simpson's rules. In [8] the present, one author has chosen the node points as the zeros of an orthogonal polynomial but, global methods are not appropriate for integrals with input functions that behave poorly in some interval $[t_j, t_j+1]$ of $[a, b]$. Therefore, for such integrals, a numerical method with no restriction on the choice of points would be more appropriate to use in order to concentrate the nodes in $[a, b]$. This is not possible with the global methods. In contrast, local methods based on piece-wise polynomial quadratures afford a flexible choice of the node points.

We give error expressions for the two methods and consequently prove their convergence and give their error bounds.

Keywords: Cauchy principal value; Lagrange interpolation; oscillation

E-mail: geokecha@unam.na

2. Preliminaries

2.1. A Well-Known Result

We define the Cauchy principal value of the integrand

$\frac{r(t)}{t-x}$ as

$$\int_a^b \frac{r(t)}{t-x}\, dt = \lim_{\varepsilon \to 0} \left\{ \int_a^{x-\varepsilon} \frac{r(t)}{t-x}\, dt + \int_{x+\varepsilon}^b \frac{r(t)}{t-x}\, dt \right\}, \qquad a < x < b \qquad (2)$$

and assume that $r(t)$ is analytic in $a < t < b$. It is well known [7] that under these conditions the integral

$$\int_a^b \frac{r(t)}{t-x} \quad (3)$$

exists for all x in $a < x < b$.

2.2. Lagrange Interpolation Formula

Given $n+1$ distinct points $x_0, x_1, ..., x_n$ in $[a, b]$, there is a polynomial $L_n(f;x)$ of degree $\le n$ that interpolates a function $f(x)$ at x_i, $i = 0, 1, ..., n$, which we may express as:

$$L_n(f;x) = \sum_{i=0}^{n} l_i(x) f(x_i) \quad (4)$$

and where the polynomials $l_i(x)$, the lagrange coefficients, are given by:

$$L_i(x) = \prod_{j \neq i}^{n} \left(\frac{x - x_j}{x_i - x_j}\right), i = 0, 1, ..., n \quad (5)$$

The proof of this important result is given by Atkinson [2, p. 132] and the interpolation error expressed as:

$$f(x) - \sum_{i=0}^{n} l_i(x) f(x_i) = \frac{(x-x_0)(x-x_1)...(x-x_n)}{(n+1)!} f^{(n+1)}(\zeta);\ a<\zeta<b \quad (6)$$

3. Methods

3.1. Linear Piecewise Polynomial Quadrature Rule

Suppose we set $h = \frac{b-a}{n}$ and define $y_r = a + rh$, $r = 0, 1, ..., n$. Then let the set $X = \{y_0, y_1, ..., y_n\}$ be the space of $n+1$ knots in $[a, b]$. Then, a linear interpolant in any interval $[y_r, y_{r+1}] \subseteq [a, b]$ which approximates $g(y)$ is given by:

$$\frac{(y - y_{r+1})}{(y_r - y_{r+1})} g(y_r) + \frac{(y - y_r)}{(y_{r+1} - y_r)} g(y_{r+1})$$

Then, substituting this for $g(y)$ in (1), we have

$$I_n(w;s) = \sum_{r=0}^{n-1} \frac{g(y_r)}{(y_r - y_{r+1})} \int_{y_r}^{y_{r+1}} \left(\frac{y - y_{r+1}}{y - s}\right) e^{iwy}\, dy$$

$$+ \sum_{r=0}^{n-1} \frac{g(y_{r+1})}{(y_{r+1} - y_r)} \int_{y_r}^{y_{r+1}} \left(\frac{y - y_r}{y - s}\right) e^{iwy}\, dy$$

$$= \frac{1}{h} \sum_{r=0}^{n-1} g(y_r) \left[\frac{i}{w}(e^{iwy_{r+1}} - e^{iwy_r}) + (y_{r+1} - s)\, q_r\right]$$

$$+ \frac{1}{h} \sum_{r=0}^{n-1} g(y_{r+1}) \left[\frac{i}{w}(e^{iwy_r} - e^{iwy_{r+1}}) + (s - y_r)\, q_r\right]$$

$$= \left(\frac{n}{b-a}\right) \sum_{r=0}^{n-1} g(y_r) \left[-pe^{i\alpha_r} + (y_{r+1} - s)\, q_r\right]$$

$$+ \left(\frac{n}{b-a}\right) \sum_{r=0}^{n-1} g(y_{r+1}) \left[-pe^{i\alpha_r} + (s - y_r)\, q_r\right]$$

Thus, after a series of simplification and rearrangement, we obtain the approximate rule

$$I_n(w;s) = \left(\frac{n}{b-a}\right) \sum_{r=0}^{n-1} [p[g(y_r + h) - g(y_r)]\, e^{i\alpha_r}$$
$$+ q_r [(y_r + h - s)\, g(y_r) + (s - y_r)\, g(y_r + h)] \quad (7)$$

where

$$p = \frac{2}{w} \sin\left(\frac{wh}{2}\right) \quad (8)$$

$$\alpha_r = \frac{w}{2}(2y_r + h) \quad (9)$$

$$q_r = \int_{y_r}^{y_{r+1}} \frac{e^{iwy}}{y - s}\, dy \quad (10)$$

A sufficient condition for the existence of q_r is that e^{iwy} is Hölder continuous in every open subinterval of $[a, b]$, a condition satisfied by the function e^{iwy}.

$$Re[q_r] = \cos ws\, C(u_{r+1}) - \sin ws S(u_{r+1}) + \sin ws S(u_r) - \cos ws C(u_r) \quad (11)$$

$$Im[q_r] = \cos ws S(u_{r+1}) + \sin ws C(u_{r+1}) - \cos ws S(u_r) - \sin ws C(u_r) \quad (12)$$

where

$$u_r = w(y_r - s); \qquad u_{r+1} = w(y_r + h - s). \quad (13)$$

C_i and S_i are the cosine and sine integrals respectively and may be expressed in the following integral forms [1, Equations 5.5.1, 5.2.27].

$$S(z) = \int_0^z \frac{\sin t}{t}\, dt, \qquad C(z) = -\int_z^\infty \frac{\cos t}{t}\, dt \quad (14)$$

These also have the series expansions [1, Equations 5.2.14, 5.2.16]

$$S(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)(2k+1)!} \quad (15)$$

$$C(x) = \gamma + \ln x + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{(2k)(2k)!} \quad (16)$$

where γ is Euler's constant.

Thus, in view of (7), (8), (9), (11), or (12), (13), (15) and (16) we may be able to approximate (1) numerically. Observe that, unlike the algorithms in [8], no modification is feasible in Equations (11) and (12) when any of the nodes coincides with s, say if $y_r = s$. For in this case, we might be required to evaluate $C_i(0)$, which is undefined. However, there is a simple remedy; vary n and hope that no other node coincides with s.

The rule (7) is simple and computationally efficient to use. In the numerical computation of S_i and C_i, we have used the truncated sums of (15) and (16) within an acceptable tolerance.

3.2. Proof of Convergence

Following [3], let

$$g_n(y) = \frac{(y-y_{r+1})}{(y_r-y_{r+1})}g(y_r) + \frac{(y-y_r)}{(y_{r+1}-y_r)}g(y_{r+1}),\ y_r \le y \le y_{r+1}$$

be the approximating sequence $\{g_n(y)\}$ to $g(y)$. Then the error can be expressed as

$$E_n(g;w;s) = \int_a^b (g-g_n)\frac{e^{iwy}}{(y-s)}dy$$

Hence,

$$|E_n(g;w;s)| \le \|g-g_n\| |\beta|$$

where

$$|\beta| = \int_a^b \left|\frac{e^{iwy}}{(y-s)}\right| dy$$

$$\le \ln\left|\frac{b-s}{a-s}\right|$$

Suppose $\|g-g_n\| \le W(g;h)$ is the modulus of continuity of g which is defined as $W(g;h) = \max |g(x_2) - g(x_1)|, |x_2 - x_1| \le h$ for any $x_1, x_2 \in [a, b]$.

Then,

$$|E_n(g;w;s)| \le W(g;h)\ln\left|\frac{b-s}{a-s}\right|$$

But, since g is continuous, $W(g;h) \to 0$ as $h \to 0$.

Thus, $|E_n(g;w;s)| \to 0$ as $n \to \infty$

Moreover, if we assume that $g(y) \in C^2[a,b]$ and use the error formula for Lagrange interpolation (6), we can obtain the following error bound.

$$|E_n(g;w;s)| = \frac{1}{2}\sum_{r=0}^{n-1}\int_{y_r}^{y_{r+1}} (y-y_r)(y-y_{r+1})g''(\zeta)\frac{e^{iwy}}{y-s}dy$$

$$\le \frac{1}{2}\sum_{r=0}^{n-1}\int_{y_r}^{y_{r+1}} |(y-y_r)(y-y_{r+1})||g''(\zeta)|\left|\frac{e^{iwy}}{y-s}\right|dy$$

$$\le \frac{h^2}{8}\sum_{r=0}^{n-1}|g''(\zeta)|\int_{y_r}^{y_{r+1}}\frac{dy}{|y-s|}$$

$$\le \frac{h^2}{8}g''(\zeta)\int_a^b\frac{dy}{|y-s|},\quad g''(\zeta) = \max_{a\le\zeta_r\le b}|g''(\zeta_r)|$$

$$\le \frac{h^2}{8}g''(\zeta)\ln\left|\frac{b-s}{a-s}\right| \quad \zeta \in (a,b)$$

3.3. Quadratic Piecewise Polynomial Quadrature Rule

Suppose we divide the interval $[a, b]$ into $2n$ equal parts. Define $y_i = a + jh, j = 0, 1, ..., 2n, h = \frac{b-a}{2n}$.

Using the Lagrange interpolation formula, the quadratic polynomial in the interval $[y_{2j-2}, y_{2j}]$, which interpolates $g(y)$ at the points y_{2j-2}, y_{2j-1}, and y_{2j} can be expressed as

$$\frac{(y-y_{2j-1})(y-y_{2j})g(y_{2j-2})}{2h^2} - \frac{(y-y_{2j-2})(y-y_{2j})g(y_{2j-1})}{h^2}$$

$$+\frac{(y-y_{2j-2})(y-y_{2j-1})g(y_{2j})}{2h^2}$$

Thus, substituting this for $g(y)$ in (1) and after a series of evaluation and simplification, we obtain the approximate rule

$$I_{2n}(w;s) = \sum_{j=1}^{n} D_{1j}\frac{g(y_{2j-2})}{2h^2} - D_{2j}\frac{g(y_{2j-1})}{h^2} + D_{3j}\frac{g(y_{2j})}{2h^2} \quad (17)$$

where

$$D_{1j} = \frac{2h}{w}\sin(wy_{2j-2}) - \frac{2}{w^2}\sin(wh)\sin(wy_{2j-1})$$
$$+\frac{2(s-y_{2j-1})}{w}\sin(wh)\cos(wy_{2j-1})$$
$$+i[\frac{2h}{w}\cos(wy_{2j-2}) + \frac{2}{w^2}\sin(wh)\cos(wy_{2j-1}) +$$
$$\frac{2(s-y_{2j-1})}{w}\sin(wh)\sin(wy_{2j-1})] + (s-y_{2j-1})(s-y_{2j})F_j$$

$$D_{2j} = \frac{2h}{w}\sin(wy_{2j-2}) - \frac{2}{w^2}\sin(wh)\sin(wy_{2j-1})$$
$$+\frac{2(s-y_{2j-2})}{w}\sin(wh)\cos(wy_{2j-1})$$
$$+i[-\frac{2h}{w}\cos(wy_{2j-2}) + \frac{2}{w^2}\sin(wh)\cos(wy_{2j-1}) +$$

$$\frac{2(s-y_{2j-2})}{w}\sin(wh)\sin(wy_{2j-1})] + (s-y_{2j-2})(s-y_{2j})F_j.$$

$$D_{3j} = \frac{2h}{w}\cos(wh)\sin(wy_{2j-1}) - \frac{2}{w^2}\sin(wh)\sin(wy_{2j-1})$$

$$+\frac{2(s-y_{2j-2})}{w}\sin(wh)\cos(wy_{2j-1})$$

$$+i[-\frac{2h}{w}\cos(wh)\cos(wy_{2j-1}) + \frac{2}{w^2}\sin(wh)\cos(wy_{2j-1}) +$$

$$\frac{2(s-y_{2j-2})}{w}\sin(wh)\sin(wy_{2j-1})] + (s-y_{2j-2})(s-y_{2j-2})F_j.$$

$$F_j = \int_{y_{2j-2}}^{y_{2j}} \frac{e^{iwy}}{y-s}dy. \qquad (18)$$

$$\mathrm{Re}[F_j] = \cos ws C_i(u_{2j}) - \sin ws S_i(u_{2j}) + \sin ws S_i(u_{2j-2}) - \cos ws C_i(u_{2j-2}) \qquad (19)$$

$$\mathrm{Im}[F_j] = \cos ws S_i(u_{2j}) + \sin ws C_i(u_{2j}) - \cos ws S_i(u_{2j-2}) - \sin ws C_i(u_{2j-2}) \qquad (20)$$

where

$$U_{2j} = w(y_{2j}-s); \quad u_{2j-2} = w(y_{2j-2}-s) \qquad (21)$$

3.4. Proof of Convergence

Let

$$g_{2n}(y) = \frac{(y-y_{2j-1})(y-y_{2j})g(y_{2j-2})}{2h^2} - \frac{(y-y_{2j-2})(y-y_{2j})g(y_{2j-1})}{h^2}$$
$$\frac{(y-y_{2j-2})(y-y_{2j-1})g(y_{2j})}{2h^2} + y_{2j} \le y \le y_{2j-2}$$

be the approximating sequence $\{g_{2n}(y)\}$ to $g(y)$.

Suppose we set the error due to the rule as

$$E_{2n}(g;w;s) = \int_a^b (g-g_{2n})\frac{e^{iwy}}{y-s}dy$$

It is shown [3] that $\|g-g_{2n}\| \le \frac{5}{4}W(g;2h)$.

Thus, in view of this result we have

$$|E_{2n}(g;w;s)| \le \frac{5}{4}W(g;2h)\ln\left|\frac{b-s}{a-s}\right|$$

Since g is continuous, $W(g;2h) \to 0$ as $h \to 0$

Furthermore, assume that $g''' \in C[a,b]$. In view of (6), we can show and obtain the error bound

$$|E_{2n}(g;w;s)| \le \frac{7}{72}h^3(b-a)\ln\left|\frac{b-s}{a-s}\right|g'''(\gamma), \gamma \in (a,b)$$

4. Two Numerical Examples

We present here two numerical examples to illustrate the two rules hitherto derived.

(a) Consider the following integral taken from [8].

$$\mathrm{Im}(10,-.13) = \int_{-1}^{1} \frac{\sin x \sin 10x dx}{x+.13}$$

Applying the quadrature rule (7) to this integral and making use of Equations (8-9, 12-15), we obtain the results:

n	$I_n(10,-.13)$
10	-.136327872536
14	-.136327866384
18	-.136327866153

Exact value: -.136327866164

(b) Again, consider from [8] the integral

$$\mathrm{Im}(12,0) = \int_{-1}^{1} \frac{e^x \sin 12x dx}{x}$$

We applied the rule (17) to this integral and obtained

n	$I_n(12,0)$
8	2.92914006215
10	2.92914005321
12	2.92914005409

Exact value: 2.92914005409

5. Conclusion

Two low-order quadrature rules have been derived for the numerical evaluation of Cauchy principal value integrals of oscillatory kind. Two numerical examples are presented to illustrate the implementation of the two rules. The convergence of the two schemes was proven and some error bounds also obtained.

6. Acknowledgements

I wish to express my gratitude to the referee for his useful comments and suggestions.

References

1. Abramowitz, M. and Stegun, I.A., *Handbook of mathematical functions, with formulas, graphs, and mathematical tables.* Dover Publications, INC, New York, (1970).
2. Atkinson, K.E. *An introduction to numerical analysis*, (2nd Edn.), John Wiley and Sons, New York, (1988).
3. Atkinson, K.E. The numerical solution of Fredholm integral equations of the second kind, *Siam J. Numer. Anal.*, 4, 3,

337-348, (1967).

4. Gerasoulis, A. Piecewise polynomial quadratures for Cauchy singular integrals, *Ibid.*, **23**, 4, 891-902, (1986).
5. Hunter, D. B. Some Gauss-type formulae for the evaluation of Cauchy principal values of integrals, *Numer. Math.*, **19**, 419-424, (1972).
6. Littlewood, R. K. and Zakian, V, Numerical evaluation of Fourier integrals, *J. Inst. Math. Appl.*, **18**, 331-339, (1976).
7. Noble, B. and Beighton, S. Error estimates for three methods of evaluating Cauchy principal value integrals, *Ibid.*, **26**, 431-446, (1980).
8. Okecha, G. E. Quadrature formulae for Cauchy principal value integrals of oscillatory kind, *Math. Comp.*, **49**, 179, 259-268, (1987).
9. Piessens, R. and Poleunis, F. A numerical method for the integration of oscillatory functions, *BIT*, **11**, 317-327, (1971).
10. Ting, B. and Luke, Y. L. Computation of integrals with oscillatory and singular integrands, *Math. Comp.*, **37**, 169-183, (1981).