J. Sci. I. R. Iran

Vol.8 No.3
Summer 1997

INITIAL RAMIFICATION INDEX OF NONINVARIANT VALUATIONS ON FINITE DIMENSIONAL DIVISION ALGEBRAS

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Abstract

Let D be a division ring with centre K and $dim_K D<\infty$, ω a valuation on K and υ a noninvariant extension of ω to D. We define the *initial ramification index* of υ over ω, $\varepsilon(\upsilon/\omega)$. Let A be a valuation ring of ω with maximal ideal m, and $\upsilon_1, \upsilon_2, \ldots, \upsilon_s$ noninvariant extensions of ω to D with valuation rings $A_1, A_2, \ldots, A_s$. If $B=\cap_{i=1}^{s} A_i$, it is shown that the following conditions are equivalent: (i) B is a finite A-module, (ii) B is a free A-module, (iii) $[B/mB: A/m]=[D:K]$, (iv) $\Sigma_{i=1}^{s} e(\upsilon_i/\omega)f(\upsilon_i/\omega)=[D:K]$ and $\varepsilon(\upsilon_i/\omega)=e(\upsilon_i/\omega)$. It is also proved that if $\varepsilon(\upsilon/\omega)=e(\upsilon/\omega)$, and any of (i) - (iv) holds, then υ is invariant.

1. Introduction

Let D be a division ring and Γ a totally ordered set. A function $\upsilon : D\rightarrow \Gamma \cup \{\infty\}$ is called a noninvariant valuation if the following conditions are satisfied:

v.1. $\upsilon(d)=\infty \Leftrightarrow d=0,$

v.2. $\upsilon(a+b)\geq min\{\upsilon(a), \upsilon(b)\},$

v.3. $\upsilon(a)\leq \upsilon(b) \Rightarrow \upsilon(ca)\leq \upsilon(cb), \quad a,b,c \in D.$

In this case Γ is called a value set. Also the set $B_\upsilon=\{d\in D|\ \upsilon(d)\geq \upsilon(1)\}$ is called a valuation ring of υ, and $m_\upsilon=\{d\in D|\ \upsilon(d)>\upsilon(1)\}$ is its maximal ideal.

Let D be a division ring, B be a valuation ring of D and $\Gamma_1=\{dB|\, d\in D\}$. Then $\{\Gamma_1, \leq\}$ is a totally ordered set with the greatest element $0B=\{0\}$,

$$dB\leq d'B \Leftrightarrow d'B\subseteq dB$$

where dB is a fractional principal right ideal of D. If we set $0B=\infty$, $\Gamma=\Gamma_1\backslash\{0B\}$ and define

$$\upsilon: D\rightarrow \Gamma \cup \{\infty\}$$

by the rule

$$d\rightarrow dB,$$

then υ is a noninvariant valuation on D, where $B_\upsilon=B$ (*cf.* [5], [3]).

Let $G_\upsilon=\{\tilde{x}\mid x\in D^*\}$, where $D^*=D\backslash\{0\}$ and $\tilde{x}$ is an order preserving bijection on Γ as follows:

$$\tilde{x}: \Gamma\rightarrow\Gamma$$

$$\upsilon(d)\rightarrow \upsilon(xd).$$

Then G_υ is a group, with respect to the composition of functions, which possesses a canonical order:

$$\tilde{x}\leq\tilde{y} \Leftrightarrow \tilde{x}(\upsilon(d))\leq\tilde{y}(\upsilon(d)),\ d\in D^*.$$

With respect to this order, G_υ is a partially ordered group.

Keywords: Initial ramification index; Noninvariant valuation

B_υ and υ are called invariant if for all $d \in D$, $dB_\upsilon = B_\upsilon d$. Therefore we can define a multiplication on Γ via

$$\upsilon(d).\upsilon(d')= \upsilon(dd').$$

Since B_υ is invariant, the multiplication is well-defined and Γ is a totally ordered group. Furthermore, G_υ is totally ordered and

$$f: \Gamma \rightarrow G_\upsilon$$

$$\upsilon(x) \rightarrow \tilde{x}$$

is an order-preserving group-isomorphism (cf. [6], [7], [8], [9], [10]). Also in this case υ is a Krull valuation on D (cf. [1], [12]).

2. Major Subsets

Let D, Γ, υ, B_υ and G_υ be as in Section 1.

Definition 2.1. Let S be a totally ordered set, then a subset M of S is called major, if for every $x \in M$ and $y \in S$

$$y \geq x \Rightarrow y \in M.$$

Let M be a major subset of Γ and N a right B_υ-module of D. Then $a(M)= \{d \in D \mid \upsilon(d) \in M \cup \{\infty\}\}$ is a right B_υ-module of D and $M(N)= \{\upsilon(x) \mid x \in N \setminus \{0\}\}$ is a major subset of Γ, and it is easily shown that there is a one to one correspondence between major subsets of Γ and $a(M)$'s in D. This correspondence is an order preserving, i.e. for two major subsets M_1 and M_2 of Γ, we have

$$M_1 \subseteq M_2 \Leftrightarrow a(M_1) \subseteq a(M_2).$$

(cf. [8, Exercise 2, Ch. 1]).

As a consequence we obtain

Corollary 2.2. (i) For any major subset T of Γ, $M(a(T))= T$. (ii) For any right B_υ-module N of D, $a(M(N))= N$.

Let Γ be as above, we put $\Gamma_+ = \{dB_\upsilon \mid d \in B_\upsilon, d^{-1} \notin B_\upsilon\}$ and M is a major subset of Γ_+. In this case, $0 \neq x \in a(M) \Rightarrow xB_\upsilon = \upsilon(x) \in M \Rightarrow xB_\upsilon \in \Gamma_+ \Rightarrow x \in B_\upsilon \Rightarrow a(M) \subseteq B_\upsilon$. Thus we have

Corollary 2.3. (i) If M is a major subset of Γ_+, then $a(M)$ is a right ideal of B_υ. (ii) If N is a right ideal of B_υ, then $M(N) = \{\upsilon(x) / x \in N \setminus \{0\}\}$ is a major subset of Γ_+.

3. Initial Remaification Index

Let D be a division ring with centre K, ω a valuation on K and υ a noninvariant extension of ω to D, with maximal ideals m_ω and m_υ, respectively. Also put $\Lambda= \{w(x)= xB_\upsilon \mid x \in K^* = K \setminus \{0\}\}$ and $\Lambda_+ = \Gamma_+|_K = \{kB_\upsilon \mid k \in K \cap B_\upsilon, k^{-1} \notin B_\upsilon\}$.

Combining the above and Corollary 2.3 one can easily show that

Proposition 3.1. There is a one to one correspondence between major subsets of Γ_+ which contain Λ_+ and the right ideals of B_υ which contain $m_\omega B_\upsilon$.

If D is finite dimensional over its centre, the above sets are finite and the rank of each of them is equal to a natural number n.

We now state the following definition which is one of the keys in this paper.

Definition 3.2. Let D be a division ring finite dimensional over its centre K. The above natural number is called the initial ramification index of υ over ω and it is denoted by $\varepsilon(\upsilon/\omega)$.

This definition coincides with the one in the commutative case.

To prove one of the main results in this section, we need to invoke results from [3] and [5].

Theorem A. [3, Th. 1]. Let D be a division ring with centre K, $[D:K]= n^2$ and B be a valuation ring of K. Then B possesses at most n noninvariant extensions in D.

Suppose $G_\omega = \{\tilde{k} \mid k \in K^*\}$, $G_\omega(\Gamma)= \{\hat{d_i} \mid d_i \in D^*\}$, where $\hat{d_i} = \{\tilde{k}\ (d_iB_\upsilon)= kd_iB_\upsilon \mid k \in K^*\}$.

In [5], each of the $\hat{d_i}$'s is called an *orbit*.

Definition B. [5, Section 4]. The number of distinct $\hat{d_i}$'s is called the ramification index of υ over ω and it is denoted by $e(\upsilon/\omega)$.

Theorem 3.3. Let D, K, υ, ω, Γ_+ and $\varepsilon(\upsilon/\omega)$ be as above.
(i) If Γ_+ does not contain the least element, then $\varepsilon(\upsilon/\omega)=1$.
(ii) If Γ_+ contains the least element such as d_0B_υ, $\Gamma' = \{d_0^tB_\upsilon \mid t \in Z\}$ and n is the number of orbits of the set of $G_\omega(\Gamma')= \{\hat{d_0}^t / t \in Z\}$, then $\varepsilon(\upsilon/\omega) =n$.
(iii) If m_υ, the maximal ideal of B_υ, is not principal, then $\varepsilon(\upsilon/\omega)=1$.

Proof. (i) Let xB_υ be any element in Γ_+ and $A= \{yB_\upsilon \in \Gamma_+ \mid yB_\upsilon < xB_\upsilon\}$. Clearly A is an infinite set. Since $e(\upsilon/\omega) < \infty$, there exist y_1B_υ, y_2B_υ in A such that $\hat{y}_1 = \hat{y}_2$, hence for some $k \in K^*$, $y_1B_\upsilon = y_2kB_\upsilon$, $y_1^{-1}y_2B_\upsilon = k^{-1}B_\upsilon$ or $y_2^{-1}y_1B_\upsilon = kB_\upsilon$.

We can assume that $y_2^{-1}y_1 \in B_\upsilon$. Since $y_1B_\upsilon < xB_\upsilon$ and $B_\upsilon < y_2B_\upsilon$, then $y_2^{-1}y_1B_\upsilon = kB_\upsilon < ky_2B_\upsilon = y_2kB_\upsilon = y_1B_\upsilon < xB_\upsilon$. Hence every major subset of Γ_+ which contains all elements of Λ_+, contains xB_υ and hence all elements of Γ_+. Therefore, by Definition 3.2, $\varepsilon(\upsilon/\omega) = 1$.

(ii) Let d_0B_υ be the least element in Γ_+. Since $e(\upsilon/\omega) < \infty$ the set $T = \{\hat{d}_0, \hat{d}_0^2, \ldots, \hat{d}_0^r, \ldots\}$ is finite, then there are positive integers r, $s (r>s)$ such that

$$\{d_0^r kB_\upsilon \mid k \in K^*\} = \hat{d}_0^r = \hat{d}_0^s = \{d_0^s kB_\upsilon \mid k \in K^*\}.$$

Thus for some $k \in K^*$, $d_0^r B_\upsilon = d_0^s kB_\upsilon$ or $d_0^{r-s} B_\upsilon = kB_\upsilon$, since $r - s > 0$, $d_0^{r-s} B_\upsilon \in \Gamma_+$ and $kB_\upsilon = d_0^{r-s} B_\upsilon \in \Lambda_+$.

This implies that there exists the least positive integer n, with $d_0^n B_\upsilon \in \Lambda_+$. Also n is the number of orbits of the action G_ω over Γ', that is the number of elements of the set of T.

Now suppose

$$M(yB_\upsilon) = \{zB_\upsilon \in \Gamma \mid yB_\upsilon \leq zB_\upsilon\}.$$

We show that $M(d_0B_\upsilon), M(d_0^2 B_\upsilon), \ldots M(d_0^n B_\upsilon)$ are the only major subsets of Γ_+ which contain Λ_+.

By Definition 2.1 $M(d_0^r B_\upsilon)$ is a major subset of Γ_+, we show that $\Lambda_+ \subseteq M(d_0^r B_\upsilon)$ for $1 \leq r \leq n$. Suppose $kB_\upsilon \in \Lambda_+$, since $d_0B_\upsilon \leq kB_\upsilon$, then $kB_\upsilon \in M(d_0B_\upsilon)$. If there exists some positive integer $1 \leq r \leq n$, such that $d_0^r B_\upsilon < kB_\upsilon < d_0^{r+1} B_\upsilon$, that is $kB_\upsilon \in M(d_0^r B_\upsilon)$ and $kB_\upsilon \notin M(d_0^{r+1} B_\upsilon)$, then $B_\upsilon < d_0^{-r} kB_\upsilon < d_0B_\upsilon$. This is a contradiction, because it is assumed that d_0B_υ is the least element of Γ_+, hence $kB_\upsilon \in M(d_0^r B_\upsilon))$ and $\Lambda_+ \subseteq M(d_0^r B_\upsilon)$, for $1 \leq r \leq n$.

Now let M' be another major subset of Γ_+ which contains Λ_+, then $M' \subseteq M(d_0B_\upsilon)$. On the other hand $d_0^n B_\upsilon \in \Lambda_+ \subseteq M'$ and hence $(d_0^n B_\upsilon) \subseteq M$. This implies that for some $1 \leq r \leq n$ and for $xB_\upsilon \in M'$

$$d_0^r B_\upsilon < xB_\upsilon < d_0^{r+1} B_\upsilon$$

or

$$B_\upsilon < d^{-r} xB_\upsilon < d_0B_\upsilon$$

and this is a contradiction too. Therefore $M' = M(d_0^r B_\upsilon)$, for some $1 \leq r \leq n$ and $\varepsilon(\upsilon/\omega) = n$.

(iii) On the contrary, suppose $\varepsilon(\upsilon/\omega) > 1$, by (i) Γ_+ contains the least element d_0B_υ, where $d_0 \in m_\upsilon$. It is easily shown that $m_\upsilon = d_0B_\upsilon$.□

It is well known that, if $F \subseteq E$ is a finite field extension, then any discrete valuation ring A of F extends to a discrete valuation ring B of E. Correspondingly we have the following result, when the field extension lies in a division ring D.

Proposition 3.4. The nontrivial valuation ω of K is discrete if and only if there exists a one to one correspondence between and Γ and a subset of integers Z.

Proof. Since $e(\upsilon/\omega) < \infty$ and $\Lambda = \omega(K^*) = \{kB_\upsilon \mid k \in K^*\}$ is isomorphic to a subgroup of Z, $\Gamma = \{dB_\upsilon \mid d \in D^*\}$ corresponds to a subset of integer Z. Now the proof is straightforward.□

We are now in a position to prove an interesting consequence of the above results as follows:

Corollary 3.5. (i) $\varepsilon(\upsilon/\omega) \leq e(\upsilon/\omega)$.

(ii) $\varepsilon(\upsilon/\omega) \mid e(\upsilon/\omega)$ if Γ_+ does not contain the least element or contains the least element d_0B_υ such that $d_0^{e(\upsilon/\omega)}B_\upsilon \in \Gamma_+$.

(iii) $\varepsilon(\upsilon/\omega) = e(\upsilon/\omega)$ if ω is discrete.

(iv) $\varepsilon(\upsilon/\omega) = 1$ if the valuation ring A of ω is of rank 1 and is nondiscrete.

Proof. (i) It is clear by Definition B and Theorem 3.3.

(ii) If Γ_+ does not contain the least element, then $\varepsilon(\upsilon/\omega) = 1$ and it contains the least element d_0B_υ, by Theorem 3.3 (ii), there is a least positive integer n such that $\varepsilon(\upsilon/\omega) = n$ and $d_0^n B_\upsilon = kB_\upsilon \in \Gamma_+$. By assumptions for some $k' \in K^*$, $k'B_\upsilon = d_0^{e(\upsilon/\omega)}B_\upsilon$. Now if n does not divide $e(\upsilon/\omega)$, then $e(\upsilon/\omega) = mn + r$, where $0<r<n$.

So $d_0^r B_\upsilon = k'^{-1} k^m B_\upsilon \in \Gamma_+$ and this is a contradiction. In any case $\varepsilon(\upsilon/\omega) \mid e(\upsilon/\omega)$.

(iii) By proposition 3.4, Γ_+ contains the least element d_0B_υ. We show that $\Gamma' = \{d_0^n B_\upsilon \mid n \in Z\} = \Gamma$. If $\Gamma' \neq \Gamma$, then there exist $dB_\upsilon \in \Gamma$ and a positive integer r such that $d_0^r B_\upsilon < dB_\upsilon < d_0^{r+1} B_\upsilon$, hence $B_\upsilon < d_0^{-r} dB_\upsilon < d_0B_\upsilon$ and this is a contradiction. Hence $\Gamma' = \Gamma$ and $G(\Gamma') = G(\Gamma)$, now by Definition B and Theorem 3.3 (ii), the proof is complete.

(iv) Since in this case the value group of ω does not contain the least positive element, neither does Γ. Hence, Theorem 3.3 (i) completes the proof.□

Let m_υ, m_ω be the maximal ideals of B_υ, B_ω respectively, then $[B_\upsilon/m_\upsilon : B_\omega/m_\omega]$ is called the residue class degree of υ over ω and it is denoted by $f(\upsilon/\omega)$.

We have the following, which is a generalization of [1] (Proposition 4, § 8.5, Ch.IV) and [4] (18.5 (a)).

Theorem 3.6. Let D be a division ring with centre K, ω a valuation on K and υ a noninvariant valuation on D

which extends ω. If B_ω, B_υ are the valuation rings corresponding to ω, υ with maximal ideals m_ω, m_υ respectively, then

$$[B_\upsilon/m_\omega B_\upsilon : B_\omega/m_\omega] = \varepsilon(\upsilon/\omega).f(\upsilon/\omega).$$

Proof. By Proposition 3.1 and Definition 3.2, $\varepsilon(\upsilon/\omega)$ is equal to the number of proper right ideals of B_υ which contain $m_\omega B_\upsilon$. The set of proper right ideals of B_υ which contains $m_\omega B_\upsilon$ forms a totally ordered set relative to the inclusion, hence $\varepsilon(\upsilon/\omega)$ is equal to the length of the right module $B_\upsilon/m_\omega B_\upsilon$ as a right B_υ module. It is well known that a right B_υ–module of length 1 is a 1–dimensional right vector space over B_υ/m_υ. Since B_υ/m_υ is a vector space over B_ω/m_ω of $f(\upsilon/\omega)$–dimension, hence a right B_υ–module of length $\varepsilon(\upsilon/\omega)$ is a vector space of dimension $\varepsilon(\upsilon/\omega)f(\upsilon/\omega)$ over B_ω/m_ω. Thus the proof is complete.□

4. The Relation $\sum_i e_i f_i = n$

Let D be a division ring finite dimensional over its centre K, ω a valuation on K with valuation ring A and $\upsilon_1, \upsilon_2, \ldots, \upsilon_s$ noninvariant extensions of ω to D. We say that A is defectless in D, if $\Sigma_{i=1}^{s} e(\upsilon_i/\omega)f(\upsilon_i/\omega) = dim_K D$ (*cf.* [11]).

To prove our next result we need the following, which is covered by [6,3.3 Satz].

Lemma 4.1. Let D be a division ring finite dimensional over its centre K, ω a valuation on K and $\upsilon_1, \ldots, \upsilon_s$ the noninvariant extensions of ω to D. If A_i is a valuation ring of $\upsilon_i (1 \le i \le s)$, $B = \cap_{i=1}^{s} A_i$ and $P_i = B \cap m_i$, where m_i is a maximal ideal of A_i. Then,
(i) $A_i = B_{p_i}$ (the localization of B at P_i).
(ii) D is a field of fraction (left and right) of B.
(iii) P_i's are the only maximal ideals of B.
Now it may be of interest to record the following result as the first application of Lemma 4.1.

Theorem 4.2. Let D be a division ring finite dimensional over its centre K, ω a valuation on K with valuation ring A and $\upsilon_1, \ldots, \upsilon_s$ the distinct noninvariant extensions of ω to D, with valuation rings $A_1, \ldots, A_s$. Assume that m is a maximal ideal of A and $B = \cap_{i=1}^{s} A_i$, then

$$[B/mB : A/m] = \sum_{i=1}^{s} \varepsilon(\upsilon_i/\omega) f(\upsilon_i/\omega).$$

Proof. We define $\psi : B/mB \to \prod_{i=1}^{s} A_i/mA_i$, by $\psi(b+mB) = (b+mA_1, \ldots, b+mA_s)$. It is clear that ψ is a homomorphism, $(mB)_{p_i} = mB_{p_i} = mA_i (1 \le i \le s)$. By [8, Theorem 7.5], $mB = mA_1 \cap mA_2 \cap \ldots \cap mA_s$, so ψ is one to one. By [2, Theorem 3.2], [8, p. 97, Corollary 6] A_i's are locally invariant, mA_i's are the right ideal of A_i's and if $(a_1+mA_1, \ldots, a_s+mA_s) \in \prod_{i=1}^{s} A_i/mA_i$, then a_i, a_j, mA_i and mA_j are compatible for $i, j \in \{1, 2, \ldots, s\}$ and there exists an $x \in D$ with $x - a_k \in A_k$ for $k = 1, 2, \ldots, s$ hence $x \in B$ and $\psi(x+mB) = (x+mA_1, \ldots, x+mA_s) = (a_1+mA_1, \ldots, a_s+mA_s)$. This shows that ψ is onto and hence ψ is an isomorphism from $B/mB \to \prod_{i=1}^{s} A_i/mA_i$. By Theorem 3.6 the proof is complete.

For the proof of the main result we need the following

Theorem C. ([5, Th. 4.1]). Let D, K, ω be as above and let $\upsilon_1, \upsilon_2, \ldots, \upsilon_s$ be all distinct noninvariant extensions of ω to D, then

$$\sum_{i=1}^{s} e(\upsilon_i/\upsilon) f(\upsilon_i/\upsilon) \le [D : k].$$

Theorem D. ([3, Th. 3]) Let D be a division ring finite dimensional over its centre K. Let A be a valuation ring of K. Assume that $A_1, A_2, \ldots, A_s$ are all of the noninvariant extensions of A to D. Then $B = \cap_{i=1}^{s} A_i$ is the integral closure of A in D.
Combining Theorem 4.2 and Theorem C, we obtain

Theorem 4.3. Let D be a division ring with centre K and $dim_K D < \infty$. Let ω be a valuation on K with valuation ring A and $\upsilon_1, \ldots, \upsilon_s$ be the distinct noninvariant extensions of ω to D, with valuation rings $A_1, \ldots, A_s$. Assume that m is a maximal ideal of A and $B = \cap_{i=1}^{s} A_i$, then the following conditions are equivalent.

(i) B is a finite *A-module*.
(ii) B is a free *A-module*.
(iii) $[B/mB : A/m] = [D : K]$.
(iv) $\Sigma_{i=1}^{s} e(\upsilon_i/\omega) f(\upsilon_i/\omega) = [D : K]$ *and* $\varepsilon(\upsilon_i/\omega) = e(\upsilon_i/\omega)$.

Proof. (i) ⇔ (ii) ⇔ (iii) is proved exactly similar to the commutative case, (cf. [4], 18.6). Now by Theorem 4.2, Theorem C, Corollary 3.5 and the equivalence of (i), (ii), (iii) the proof is complete.□

In view of Theorem 4.3 an interesting and important case is $\varepsilon(\upsilon_i/\omega) = e(\upsilon_i/\omega)$ in which υ_i must be invariant. So, we show the following consequence for which we are indebted to J. Gräter.

Theorem 4.4. If the conditions of Theorem 4.3 hold, then ω uniquely extends to an invariant valuation to D.

Proof. Let υ be an extension of ω to D and $\varepsilon(\upsilon/\omega) = e(\upsilon/\omega) = n > 1$. Let $m_\omega B_\upsilon \subseteq b_1 B_\upsilon \subset b_2 B_\upsilon \subset \ldots \subset b_n B_\upsilon = m_\upsilon$ be the complete chain of right ideals, where b_i 's are the distinct orbits of υ over ω. Then each d in D can be written as $d= b_i ku$, where $k \in K^*$, and u is a unit in B_υ [5]. On the contrary, assume that υ is not the only extension of ω to D. Let R be the subring of D minimal with the property of containing all extensions of ω to D [3, Lemma 4]. Then $J(R) \subset m_\omega B_\upsilon \subset B_\upsilon \subset R$, i.e. each b_i is a unit in R. With the same notations as in [3, Lemma 5], the automorphism of S induced by the inner automorphism of D which is induced by $d= b_i ku$ is the identity, i.e. $S= (R\cap K)/(N\cap K)$ and Z is purely inseparable over S, where N is the maximal ideal of R and Z is the centre of R/N. This is a contradiction as in the proof of [3, Lemma 5(i)]. Thus υ is invariant and the only extension of ω to D. So the proof is complete.□

Remark 1. Let $K \subseteq L$ be a finite separable extension, A a discrete valuation ring of K, then the conditions of Theorem 4.3 holds, but when E lies in a division ring D, this conclusion does not hold.

Example 4.5. Let $H = \left(\frac{-1\ ,\ -1}{Q}\right)$ be the usual quaternion algebra, then the *p–adic* valuation $(p>3)$ of Q does not extend to H. So by Theorem 4.4, this conclusion does not hold for Q and H.

Finally, as an immediate consequence of Theorem 4.3 and Corollary 3.5, we obtain

Corollary 4.6. For any non discrete valuation ring A of rank 1, the equivalent conditions of Theorem 4.3 hold if and only if A is defectless in D and $e(\upsilon_i/\omega)= 1$ for all $1 \leq i \leq s$.

Remark 2. To observe some of the results developed here see the example of [3, Sec. 4] or [5, Sec. 5]. Unfortunately, an example which may demonstrate all of the results can not easily be constructed. It would be an interesting exercise to think of such examples.

Acknowledgements

The author would like to thank Professors J. Gräter and M. Mahdavi-Hezavehi for their valuable suggestions and encouragement. My thanks also go to the Research Council of Sharif University of Technology for its support.

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