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# A NORM INEQUALITY FOR CHEBYSHEV CENTRES

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## Abstract

In this paper, we study the Chebyshev centres of bounded subsets of normed spaces and obtain a norm inequality for relative centres. In particular, we prove that if $T$ is a remotal subset of an inner product space $H$, and $F$ is a star-shaped set at a relative Chebyshev centre $c$ of $T$ with respect to $F$, then $\|x - q_T(x)\|^2 \geq \|x-c\|^2 + \|c-q_T(c)\|^2$ $x \in F$, where $q_T : F \to T$ is any choice function sending $x$ to the point $q_T(x)$ with $\|x - q_T(x)\| = \sup_{t\in T} \|x - t\|$ (note that $T$ is called remotal if such a choice function $q_T$ exists). We then use such an inequality to show that, under some restrictions, a uniquely remotal set is a singleton. Further, we show that if $c$ is a centre of a remotal subset $T$ of a normed space $E$ and $x \in E$, then there exists a functional $f \in E^*$ such that $\|f\| \leq 1$ and $\|x - q_T(x)\|^2 \geq \|c - q_T(c)\|^2 + 2|f(x - c)|^2 - \|x - c\|^2$.

## Introduction

Following the notation used in [6] and [7], a centre (or Chebyshev centre) of a bounded nonempty set $T$ in a normed space $E$ is an element $c$ in $E$ such that $\sup_{t\in T} \|c - t\| = \inf_{x\in E} \sup_{t\in T} \|x - t\|$. The number $\sup_{t\in T} \|c - t\|$ is called the Chebyshev radius of $T$ and is denoted by $r(T)$. The set-valued map $Q_T$ defined by $Q_T(x) = \{s \in T : \|x - s\| = \sup_{t\in T} \|x - t\|\}$, is called the farthest point map of $T$. If, for any $x$ in $E$, the set $Q_T$ is not empty (resp. is singleton), then $T$ is said to be remotal (resp. uniquely remotal). We assume for the rest of the paper that $T$ is remotal and $q_T : E \to E$ is a choice function such that $q_T(x) \in Q_T(x) \forall x \in E$. Suppose $c$ is a centre of $T$, then we have $r(T) = \sup_{t\in T} \|c - t\| = \|c - q_T(c)\| = \inf_{x\in E} \|x - q_T(x)\|$. We recall that any point $c$ in a subset $F$ of a normed space $E$ for which $\|c - q_T(c)\| = \inf\{\|x - q_T(x)\| : x \in F\}$, is called a relative Chebyshev centre of $T$ with respect to $F$. The number $\|c - q_T(c)\|$ is called the relative Chebyshev radius of $T$ with respect to $F$ and denoted by $r_F(T)$.

Let $x \in E$ and $M$ be a subset of $E$. An element $y \in M$ is said to be a best approximation to $x$ from $M$ if $\|x - y\| = \inf\{\|x - m\| : m \in M\}$. If $M$ admits a unique best approximation to every $x \in E$, then $M$ is called a Chebyshev set. Any closed convex set in a Hilbert space is a Chebyshev set. The converse is an open problem. It follows from Mazur-Tychonoff theorems that a compact uniquely remotal set in a normed space is a singleton [5]. The following well-known question is not solved yet: If $T$ is a uniquely remotal subset of a normed space, then can we conclude that $T$ is a singleton? There are some affirmative answers to this question in [1], [2] and [3]. An affirmative solution to this problem implies that every Chebyshev set in a Hilbert space is convex [5]. Using the idea of Chebyshev centre one can prove that if the farthest point map of a uniquely remotal subset $T$ of a Banach space is continuous, then $T$ is a singleton [1], [8].

**Keywords:** Uniquely remotal set; Inner product space; Chebyshev centre; Farthest point; Best approximation

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For elements $x$ and $y$ in a complex vector space $E$, we define the line segments $(x, y]$ and $[x, y]$ by $(x, y] = \{\alpha x + (1-\alpha) y : 0 \leq \alpha < 1\}$ and $[x, y] = \{\alpha x + (1-\alpha) y : 0 \leq \alpha \leq 1\}$. A subset $F$ of $E$ is said to be star-shaped at a vertex $s$ if and only if for any $x \in F$ the line segment $(s, x]$ is contained in $F$. We denoted by $E^*$ the dual of normed space $E$, that is the set of all continuous linear functionals on $E$. A duality mapping on $E$ is any mapping $D : E \rightarrow E^*$ such that for each $x \in E$, $\|D(x)\| = \|x\|$ and $< x, D(x) > = \|x\|^2$. By the Hahn-Banach theorem, such a mapping always exists.

In this paper, we study the Chebyshev centres of bounded subsets of normed spaces. We show that if $T$ is a remotal subset of an inner product space $H$, and $F$ is a star-shaped set at a relative Chebyshev centre $c$ of $T$ with respect to $F$, then $\|x - q_T(x)\|^2 \geq r_F(T) + \|x - c\|^2$ $(x \in F)$. The uniqueness of the centre follows easily from this inequality (we notice that in a uniformly rotund Banach space, every bounded subset has a unique Chebyshev centre [4]). An extension of such inequality to general normed spaces is discussed. In particular, we show that for any $x$ in $E$ and $0 < \varepsilon \leq 1$, there exists a linear functional $f$ in $E^*$ such that $\|f\| = 1$ and $\|x - q_T(x)\|^2 \geq r^2(T) + (1-\varepsilon)|f(x-c)|^2 + (\varepsilon - 1)\|x - c\|^2$. Such an inequality is used to show that, under some restrictions, a uniquely remotal set is a singleton. We also prove that the defining inequality related to the duality mapping as defined in [9] is valid for farthest point map. We denote the diameter of a bounded subset $T$ by $d(T)$, that is $d(T) = \sup\{\|t - s\| : t, s \in T\}$. We also obtain the diameter inequality $d(T) \geq \sqrt{2} r_F(T)$ for a star-shaped subset $F$ of an inner product $H$. Throughout this paper, all vector spaces are defined over the field of complex numbers, and we denote the real part of a complex number $\lambda$ by $Re\lambda$.

## Results

**Theorem 2.1.** Suppose $T$ is a remotal subset of an inner product space $(H, <.,.>)$ and $F$ is a star-shaped subset of $H$ at a vertex $c$ such that $c$ is also a relative centre of $T$ with respect to $F$. Then

*(i)* $Re < c - x, c - q_T(x) > \leq 0$ $(x \in F)$, and

*(ii)* If $q_T(c) \in F$ is a cluster point of $\cup\{Q_T(x) : x \in [c, q_T(c)]\}$, then $T$ is a singleton.

**Proof.** *(i)* Replacing $T$ by $c - T$ and $F$ by $c - F$, one can assume that $c = 0$. Let $0 < \alpha \leq 1$. By definition of the farthest point map we have

$$\|x - q_T(x)\|^2 \geq \|x - q_T(\alpha x)\|^2 \text{ and}$$
$$\|\alpha x - q_T(\alpha x)\|^2 \geq \|\alpha x - q_T(x)\|^2.$$

Hence, expanding the two inequalities and adding them together yields

$$Re < x, q_T(x) > \leq Re < x, q_T(\alpha x) > (x \in F). \quad (1)$$

We have

$$\|\alpha x - q_T(\alpha x)\|^2 \geq \|q_T(0)\|^2 \geq \|q_T(\alpha x)\|^2 (x \in F), \quad (2)$$

since $\|q_T(0)\| = \sup\{\|t\| : t \in T\}$, $q_T(\alpha x) \in T$ and $0$ is a relative centre of $T$.
We obtain from (1) and (2)

$$2Re < x, q_T(x) > \leq 2Re < x, q_T(\alpha x) > \leq \alpha\|x\|^2 \ (x \in F, 0 < \alpha \leq 1).$$

Hence $Re < x, q_T(x) > \geq 0 (x \in F)$.

*(ii)* Suppose there exists a sequence $\{\alpha_n\}$ in $[0, 1]$ such that $y_n = q_T(x_n) \rightarrow q_T(c)$, where $x_n = \alpha_n c + (1 - \alpha_n) q_T(c)$. It follows from *(i)* that $Re < c - x_n, c - y_n > \leq 0$, and hence $Re < c - q_T(c), c - y_n > \leq 0 (n = 1, 2, ...)$. The continuity of inner product implies that $\|c - q_T(c)\| \leq 0$. Therefore, $0 = \|c - q_T(c)\| = \sup\{\|c - t\| : t \in T\}$, and thus $T = \{c\}$. □

**Corollary 2.2.** Let $c$ be a Chebyshev centre of a uniquely remotal subset of an inner product space $H$. Suppose that the range of $q_T$ restricted to the line segment $[c, q_T(c)]$ admits $q_T(c)$ as a cluster point. Then $T$ is a singleton. Therefore, $T$ is a singleton if and only if the restriction of $q_T$ to $[c, q_T(c)]$ is continuous at $c$. □

**Theorem 2.3.** Suppose $T$ is a remotal subset of an inner product space $H$ and $F$ is a star-shaped subset of $H$ at a vertex $c$ such that $c$ is a relative centre of $T$ with respect to $F$. Then the following assertions are true.

$$(i)\ \|x - q_T(x)\|^2 \geq \|x - c\|^2 + r^2_F(T) \ (x \in F). \quad (3)$$

*(ii)* $c$ is unique, and if further $F \cap Q_T(c) \neq \phi$, then $d(T) \geq \sqrt{2} r_F(T) \geq \sqrt{2} r(T)$.

In fact we have $\|q_T(c) - q_T(x)\| \geq \sqrt{2} r_F(T)$, $(x \in (c, q_T(c)], q_T(c) \in F)$.

*(iii)* Suppose $T$ is uniquely remotal. If $Re < c - x_0, c - q_T(x_0) > = 0$ for some element $x_0$ in $F$, then $q_T(x_0) = q_T(c)$. Therefore, if $q_T(c) \in F$, then $T$ is a singleton if and only if $Re < c - q_T(c), c - q_T(q_T(c)) > = 0$.

**Proof.** *(i)* Replacing $T$ by $c - T$ and $F$ by $c - F$, one can assume without loss of generality that $c = 0$. Assertion *(i)* of Theorem 2.1 shows that $Re < x, q_T(x) > \leq 0$, for all $x \in F$. Hence, $Re < x, q_T(\alpha x) > \leq 0$, $(0 < \alpha \leq 1)$, since $\alpha x \in F$. Therefore,

$$r^2{}_F(T) \leq \|\alpha x - q_T(\alpha x)\|^2 = (1 - \alpha)^2 \|x\|^2 + \|x - q_T(\alpha x)\|^2 + 2(1 - \alpha) Re < x, q_T(ax) - x >$$
$$= \|x - q_T(\alpha x)\|^2 - (1 - \alpha^2)\|x\|^2 + 2(1 - \alpha) Re < x, q_T(x) >$$
$$\leq \|x - q_T(x)\|^2 - \|x\|^2 + \alpha^2\|x\|^2.$$

Thus, $\|x - q_T(x)\|^2 \geq r^2{}_F(T) + \|x\|^2 - \alpha^2\|x\|^2$, $(\alpha \in (0, 1])$.
Hence, $\|x - q_T(x)\|^2 \geq r^2{}_F(T) + \|x\|^2$, $(x \in F)$.

*(ii)* The uniqueness of $c$ follows immediately from assertion *(i)* and the definition of relative Chebyshev radius. If we put $x = q_T(c)$ in the inequality (3), we obtain

$$\|q_T(c) - q_T(q_T(c))\|^2 \geq \|c - q_T(c)\|^2 + r^2{}_F(T) = 2r^2{}_F(T),$$

hence, $d(T) \geq \|q_T(c) - q_T(q_T(c))\| \geq \sqrt{2}r_F(T) \geq \sqrt{2}r(T)$. In fact, if $0 \leq \alpha < 1$, let $x_\alpha = \alpha(c - q_T(c) + q_T(c)$. Then $x_\alpha \in F$ and

$$\|q_T(c) - q_T(x_\alpha)\|^2 = \|x_\alpha - q_T(x_\alpha) - \alpha(c - q_T(c))\|^2$$
$$= \|x_\alpha - q_T(x_\alpha)\|^2 + \alpha^2\|c - q_T(c)\|^2$$
$$- 2\alpha Re < c - q_T(c), x_\alpha - q_T(x_\alpha) >$$
$$= \|x_\alpha - q_T(x_\alpha)\|^2 + \alpha^2\|c - q_T(c)\|^2$$
$$+ 2\alpha(1 - \alpha)\|c - q_T(c)\|^2$$
$$- 2\alpha Re < c - q_T(c), c - q_T(x_\alpha) >$$

By (3) $\geq r^2{}_F(T) + \|c - x_\alpha\|^2 + \alpha(2 - \alpha)r^2{}_F(T) - 2\alpha Re < c - q_T(c), c - q_T(x_\alpha) >$
$= 2r^2{}_F(T) - 2\alpha Re < c - q_T(c), c - q_T(x_\alpha) >$.

In Theorem 2.1 *(i)* let $x = x_\alpha$ and conclude that

$$Re < c - q_T(c), c - Q_T(x_\alpha) > \leq 0, (x_\alpha \in (c, q_T(c)]). \quad (4)$$

It follows therefore that

$$\|q_T(c) - q_T(x_\alpha)\|^2 \geq 2r^2{}_F(T)$$
$$- 2\alpha Re < c - q_T(c), c - q_T(x_\alpha) > \geq 2r^2{}_F(T). \quad (5)$$

*(iii)* By the inequality (3) and the definition of the farthest point map we have $\|c - q_T(c)\|^2 \leq \|x_0 - q_T(x_0)\|^2 - \|x_0 - c\|^2 = \|c - q_T(x_0)\|^2 \leq \|c - q_T(c)\|^2$. Hence, $\|c - q_T(c)\| = \|c - q_T(x_0)\|$. Thus, $q_T(x_0) = q_T(c)$, since $T$ is uniquely remotal. If $x_0 = q_T(c)$, then $q_T(x_0) = x_0$ that is $T = \{x_0\}$.□

**Corollary 2.4.** Suppose $T$ is a uniquely remotal subset of an inner product space $H$ and $c$ is a Chebyshev centre of $T$. Then the following assertions are true:

*(i)* $\|x - q_T(x)\|^2 \geq \|x - c\|^2 + r^2(T)$

*(ii)* If $T$ is not a singleton, then $d(T) \geq \sqrt{2}r(T)$.

**Proof.** *(i)* follows immediately from the assertion *(i)* of Theorem 2.3.
*(ii)* Let $x$ be an element in $(c, q_T(c)]$. Then $Re < c - q_T(c), c - q_T(x) > \leq 0$ by (4). We show that $Re < c - q_T(c), c - q_T(x) > \neq 0$. If $Re < c - q_T(c), c - q_T(x) > = 0$, then we have $Re < c - x, c - q_T(x) > = 0$, since $x = \alpha(c - q_T(c)) + q_T(c)$ for some $\alpha \in [0, 1]$. Assertion *(iii)* of Theorem 2.3 shows that $T$ is a singleton; a contradiction. Therefore, $Re < c - q_T(c), c - q_T(x) > < 0$. We obtain from (5)

$$d^2(T) \geq \|q_T(c) - q_T(x)\|^2 \geq 2r^2(T)$$
$$- 2\alpha Re < c - q_T(c), c - q_T(x) >.$$

Hence $d(T) \geq \sqrt{2}r(T)$.□

**Theorem 2.5.** Let $E$ be a normed space and $T$ be a uniquely remotal subset of $E$. Suppose $c$ is an element of $E$ and define $E_c = \{x \in E : \|x - q_T(x)\| \geq \|c - q_T(c)\|\}$. It is obvious that $c$ is a relative Chebyshev centre of $T$ with respect to $E_c$. If, further, $c$ is the only relative Chebyshev centre of $T$ (that is $x \neq c$ implies $\|x - q_T(x)\| \neq \|c - q_T(c)\|$), then
*(i)* $E = E_c \cup [c, q_T(c)]$, and
*(ii)* $c$ is the centre of $T$ if and only if $[c, q_T(c)] \subseteq E_c$.

**Proof.** *(ii)* follows immediately from the assertion *(i)*.

*(i)* Let $\acute{E}_c$ be the complement of $E_c$ in $E$, that is $\acute{E}_c = \{x \in E: \|x - q_T(x)\| < \|c - q_T(c)\|\}$. It is enough to show that $\acute{E}_c \subseteq [c, q_T(c)]$. Let $x_0$ be an element in $\acute{E}_c$. If $x_0 = q_T(x_0)$, then $T$ is a singleton and $x_0 = q_T(c) \in [c, q_T(c)]$. Thus, we can assume $x_0 \neq q_T(x_0)$ and $c \neq q_T(c)$. Define $\alpha$ and $y_0$ by

$$\alpha = \frac{\|x_0 - q_T(x_0)\|}{\|c - q_T(c)\|}, y_0 = q_T(x_0) + \frac{1}{\alpha}(x_0 - q_T(x_0)). \quad (6)$$

Then we have $\|c - q_T(c)\| = \|y_0 - q_T(x_0)\| \le \|y_0 - q_T(y_0)\|$,

$\|y_0 - x_0\| = \|c - q_T(c)\| - \|x_0 - q_T(x_0)\|$ and

$\|c - q_T(c)\| = \|y_0 - q_T(x_0)\| \le \|y_0 - q_T(y_0)\|$

$\le \|y_0 - x_0\| + \|x_0 - q_T(y_0)\|$

$\le \|y_0 - x_0\| + \|x_0 - q_T(x_0)\| = \|c - q_T(c)\|.$

Hence, $\|c - q_T(c)\| = \|y_0 - q_T(x_0)\| = \|y_0 - q_T(y_0)\|$. Thus, $c = y_0$ and $q_T(x_0) = q_T(c)$, since $c$ is unique and $T$ is uniquely remotal. It follows therefore from (6) that $x_0 = \alpha c + (1 - \alpha)q_T(c)$ is an element of $[c, q_T(c)]$.□

It is not known whether the inequality (3) in Theorem 2.3 can be extended to an arbitrary normed space $E$. However, we can prove a more or less similar inequality using the continuous linear functionals in $E^*$.

**Theorem 2.6.** Let $c$ be a centre of a remotal subset $T$ of a normed space $E$. Then the following properties hold:

*(i)* Given $x$ in $E$ and $0 < \varepsilon \le 1$, there exists an element $f$ in $E^*$ such that $\|f\| = 1$ and $\|x - q_T(x)\|^2 \ge r^2(T) + (1 - \varepsilon)|f(x) - f(c)|^2 + (\varepsilon - 1)\|x - c\|^2$.

*(ii)* For each $x$ in $E$, there exists an element $f$ in $E^*$ such that $\|f\| \le 1$ and $\|x - q_T(x)\|^2 \ge r^2(T) + 2|f(x - c)|^2 - \|x - c\|^2$.

**Proof.** *(i)* We may and shall assume that $c = 0$. By the Hahn-Banach theorem, there exists a functional $f$ in $E^*$ such that $\|f\| = 1$ and

$f(q_T(\varepsilon x) - \varepsilon x) = \|\varepsilon x - q_T(\varepsilon x)\|$. Note that

$\|q_T(0)^2 \le \|\varepsilon x - q_T(\varepsilon x)\|^2 = |f(q_T(\varepsilon x) - \varepsilon x)|^2$

$= |f(q_T(\varepsilon x))|^2 + |f(\varepsilon x)|^2 - 2\varepsilon Re f(x)\,\overline{f(q_T(\varepsilon x))}$

$\le \|q_T(\varepsilon x)\|^2 + \varepsilon^2\|x\|^2 - 2\varepsilon Re f(x)\,\overline{f(q_T(\varepsilon x))}$

$\le \|q_T(0)\| + \varepsilon^2\|x\|^2 - 2\varepsilon Re f(x)\,\overline{f(q_T(\varepsilon x))}$

Hence

$$2Re f(x)\,\overline{f(q_T(\varepsilon x))} \le \varepsilon\|x\|^2. \quad (7)$$

Observe that

$\|q_T(0)\|^2 \le \|\varepsilon x - q_T((\varepsilon x)\|^2 = |f(q_T(\varepsilon x) - x + x - \varepsilon x)|^2$

$\le \|x - q_T(\varepsilon x)\|^2 + (1 - \varepsilon)^2\|x\|^2 + 2(1 - \varepsilon) Re f(x)\,\overline{f(q_T(\varepsilon x))} - 2(1 - \varepsilon)|f(x)|^2.$

It follows therefore from (7) that

$r^2(T) = \|q_T(0)\|^2 \le \|x - q_T(\varepsilon x)\|^2 + (1 - \varepsilon)^2\|x\|^2 + \varepsilon(1 - \varepsilon)\|x\|^2 - 2(1 - \varepsilon)|f(x)|^2.$

*(ii)* Fix $x$, $q_T(x)$ and take $\varepsilon = \frac{1}{n}$ for each $n$. By the assertion *(i)* there exist linear functionals $f_n$ in $E^*$ such that $\|f_n\| = 1$ and

$$\|x - q_T(x)\|^2 \ge r^2(T) + 2(1 - \frac{1}{n})\|f_n(x - c)\|^2 + (\frac{1}{n} - 1)\|x - c\|^2. \quad (8)$$

By the Alaoglu-Banach theorem, $\{f_n\}$ admits a $W^*$-cluster point [10]. Take $f$ to be a $W^*$-cluster point of $\{f_n\}$ and assume without loss of generality that $f_n \xrightarrow{W^*} f$. In (8), letting $n \to \infty$ yields

$\|x - q_T(x)\|^2 \ge r^2(T) + 2\|f(x - c)\|^2 - \|x - c\|^2.$□

The following theorem also shows that a similar defining inequality, as defined in [9], is valid for farthest point maps (antiprojections). This theorem does not really belong here, but there seems to be no better spot for it.

**Theorem 2.7.** Let $T$ be a remotal subset of a normed space $E$ and $D$ be duality mapping on $E$. Then for each $x \in E$ *and* $t \in T$

*(i)* $Re < t - q_T(x), D(q_T(x) - x) > \le 0$

*(ii)* $Re < 2x - t - q_T(x), D(q_T(x) - x) > \le 0$

*(iii)* If $c$ is a centre of $T$ then $Re < t - c + x - q_T(x), D(q_T(x) - x) > \le 0$, and therefore $Re < x - c, D(q_T(x) - x) > \le 0$.

**Proof.** *(i)* and *(ii)* are immediate consequences of the following

$Re < \pm(t - x), D(q_T(x) - x) > \le \|t - x\|\,\|x - q_T(x)\| \le \|x - q_T(x)\|^2.$

*(iii)* If $c$ is a centre of $T$, then for each $t \in T$ we have

$Re < t - c, D(q_T(x) - x) > \le \|t - c\|\,\|x - q_T(x)\| \le \|c - q_T(c)\|\,\|x - q_T(x)\|$

$$\leq \|x - q_T(x)\|^2 = < q_T(x) - x, D(q_T(x) - x) >$$

Hence

$$Re < t - c - q_T(x) + x, D(q_T(x) - x) > \leq 0. \quad (9)$$

Put $t = q_T(x)$ in (9), we get $Re < x - c, D(q_T(x) - x) > \leq 0$. □

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