ON THE LIFTS OF SEMI-RIEMANNIAN METRICS

N. Broojerdian

Department of Mathematics, Faculty of Science, University of Tehran, Tehran, Islamic Republic of Iran

Abstract

In this paper, we extend Sasaki metric for tangent bundle of a Riemannian manifold and Sasaki-Mok metric for the frame bundle of a Riemannian manifold [1] to the case of a semi-Riemannian vector bundle over a semi-Riemannian manifold. In fact, if E is a semi-Riemannian vector bundle over a semi-Riemannian manifold M, then by using an arbitrary (linear) connection on E, we can make E, as a manifold, into a semi-Riemannian manifold. When the metric of the vector bundle E is parallel with respect to the chosen connection, we compute the Levi-Civita connection of E, its geodesics, and its curvature tensors. We also show that the sphere and pseudo-sphere bundles of E are non-degenerate submanifolds of E, and we shall compute their second fundamental forms. We shall also prove some results on the metric of E.

Preliminaries

Let $(V,.)$ be a finite dimensional inner product space. For each $v \in V$, $sgn(v)$ is defined as follows

$$sgn(v) = \begin{cases} +1 & v.v > 0 \\ 0 & v.v = 0 \\ -1 & v.v < 0 \end{cases}$$

There exist bases like $\{e_1, ..., e_n\}$ in V such that $e_i . e_j = \pm \delta_{ij}$ [4]. We call them orthonormal bases. Vector spaces associated with V such as V^*, $L(V)$, $\otimes^r V$, and $\wedge^r V$ can be made into inner product spaces in a natural way. If the inner product is not positive or negative definite, the restriction of it to a subspace is not in general an inner product. In fact, if W is a subspace of V, and $W^\perp$ is its orthogonal, the restriction of the inner product to W makes it an inner product space if and only if $W \cap W^\perp = \{0\}$ (or equivalently $V = W \oplus W^\perp$). If this is the case we call W a non-degenerate subspace.

Keywords: Semi-Riemannian; Vector field; Horizontal vertical; Curvature

Let $\diamondsuit(V)$ denote the space of all antisymmetric linear maps on V, the linear map $\wedge^2 V \to \diamondsuit(V)$, which sends $u \wedge v$ to the map $x \mapsto (v.x)u - (u.x)v$, is an isomorphism of inner product spaces. We shall identify $\wedge^2 V$ and $\diamondsuit(V)$ under this isomorphism.

By manifolds we mean C^∞ real manifolds. The vector bundle (E, π, M, F) will be denoted by $E \xrightarrow{\pi} M$, and the fiber over $p \in M$ will be denoted by E_p. VE will denote the vertical bundle of E. It is well known that VE is a subbundle of TE [9]. For $\xi, \eta \in E$ with $\pi(\xi) = \pi(\eta)$ we set $I_\xi \eta = \frac{d}{dt}|_{t=0} (\xi + t\eta)$. Clearly $I_\xi \eta \in (VE)_\xi$, and it is called the vertical lift of ξ at η.

To each connection ∇ on E there correspond a horizontal subbundle $\mathcal{H}$ (of TE), a connection map k: $TE \to E$, and a parallel system IP [9]. Let $p \in M$, $u \in T_pM$ and $\xi \in E_p$. There exists a unique vector on $\mathcal{H}_\xi$ such that its image under $d\pi$ is u. This vector is called the horizontal lift of u at ξ, and it will be denoted by $\overline{u}_\xi$. The set of all sections of a vector bundle $E \xrightarrow{\pi} M$ will be denoted by ΓE.

Let E be a semi-Riemannian vector bundle over M. The vector bundles E^* (dual of E), $L(E)$, $\otimes^r E$, $\wedge^r E$ ($1 \leq r$) can be made into semi-Riemannian vector bundles in a natural way.

Let M be a semi-Riemannian manifold, a submanifold N of M is called semi-Riemannian submanifold, if for each $p \in N$, T_pN is a non-degenerate subspace of T_pM (of course, if N is connected, then the signature of the inner product on each fiber is constant.). Clearly, in this case, the restriction of the metric of M to N makes it into a semi-Riemannian manifold. Let ∇^M, ∇^N denote the Levi-Civita connection of M and N, respectively, and E be the restriction of TM on N (or equivalently, E be the pull-back of TM over the inclusion map $N \xrightarrow{i} M$). The pull-back of ∇^M, which is a connection on E will be denoted by the same symbol ∇^M. Let p_1: $E \rightarrow TN$ be the orthogonal projection. Then for each U, $V \in \chi N \subseteq \Gamma E$

$$\nabla^N_U V = p_1 (\nabla^M_U V)$$

[8]. Let $TN^\perp$ be the orthogonal complement of the vector bundle TN in E, and p_2: $E \rightarrow TN^\perp$ be the orthogonal projection. The map $\pi : TN \otimes TN \rightarrow TN^\perp$ which is defined by $\pi(U, V) = p_2 (\nabla^M_U V) = \nabla^M_U V - \nabla^N_U V$ is a symmetric tensor, called second fundamental form of N[8]. Knowing π, we can compute different curvatures of N in terms of the corresponding curvatures of M. Let R^M and R^N denote the curvature tensors of M and N, respectively. Then for each U, V, W, $P \in \chi N$ we have
$< R^N (U, V)(W), P > = < R^M (U, V)(W), P > + < \pi(U, P), \pi(V, W) > - < \pi(V, P), \pi(U, W) >$
[8]

Let $TN^\perp$ be a line bundle admitting a section Z such that $< Z, Z > = \pm 1$. Then the second fundamental form of N determines a symmetric bilinear form $\overline{\pi}$ on TN as follows:

$$U, V \in \chi N \qquad \overline{\pi}(U, V) = < \pi(U, V), Z >$$

The bilinear form $\overline{\pi}$ in turn determines a self-adjoint bundle map $S : TN \rightarrow TN$

$$U, V \in \chi N \qquad < S(U), V > = \overline{\pi}(U, V).$$

The bundle map S is called Weingarten map of N (with respect to Z), and can be computed directly as follows:

$$U \in \chi N \qquad S(U) = -\nabla^M_U Z$$

[8]. Knowing S, the second fundamental form π can be computed as follows:

$$U, V \in \chi N \qquad \pi(U, V) = < S(U), V > sgn(Z) Z$$

Fundamental Vector Fields of a Vector Bundle

Assume that $\pi : E \rightarrow M$ is vector bundle. Let (x, U) be a chart of M, and $\pi^{-1}(U) \xrightarrow{(\pi, \psi)} U \times IR^k$ be a trivialization of E over U. If $x = (x^1, ..., x^n)$ and $\psi = (\psi^1, ..., \psi^k)$, then

$$(x^1 \circ \pi, ..., x^n \circ \pi, \psi^1, ..., \psi^k)$$

is a chart of E whose domain of definition is $\pi^{-1}(U)$. For each $p \in U$, the restriction of ψ^m ($m = 1, 2, ..., k$) to E_p belongs to E_p^*. Let $\overline{x}^i = x^i \circ \pi$ for $i = 1, 2, ..., n$.

Clearly the vector fields $\frac{\partial}{\partial \psi^m}$ ($m = 1, 2, ..., k$) generate the vertical subbundle of $TE|_{\pi^{-1}(U)}$.

A map $F : E \rightarrow E$ is called a strong bundle map if every fiber E_p ($p \in M$) is invariant under F. If restriction of F to each E_p is linear, it is called a linear strong bundle map.

To each strong bundle map $F : E \rightarrow E$ (not necessarily linear) there corresponds a vertical vector field of E (a section of vE) which will be denoted by $\widetilde{F}$ and is defined by

$$\xi \in E \qquad \widetilde{F}_\xi = l_\xi F(\xi)$$

$\widetilde{F}$ is smooth, because in local coordinate systems defined above, if $\{\psi_1, ..., \psi_k\}$ is dual of $\{\psi^1, ..., \psi^k\}$, and F is expressed as $F(\xi) = f^m(\xi)\psi_m$, then f^m 's are smooth ($f^m = \psi^m \circ F$), and by a direct computation we have $\widetilde{F}_\xi = f^m(\xi) \frac{\partial}{\partial \psi^m}$.

For example, if $F = 1_E$, then $\widetilde{1_E}$ is the radial vector field on E. The set of all vertical vector fields on E as well as the set of all strong bundle maps on E, are modules over $C^\infty(E)$. From the definition of $\widetilde{F}$ and the local representations of F and $\widetilde{F}$, we see that the map $F \mapsto \widetilde{F}$ is a linear isomorphism between the above modules.

Let ∇ be a connection on E throughout the paper, the horizontal subbundle of E (respectively, its connection map, and its parallel system) will be denoted by $\mathcal{H}$ (respectively by k and IP).

To each strong bundle map $A : E \to TM$ (not necessarily linear) there corresponds a horizontal vector field on E (a section of $\mathcal{H}$) which will be denoted by $\bar{A}$ and is defined by

$$\xi \in E \qquad \bar{A}_\xi = \overline{A(\xi)}_\xi$$

To prove smoothness of $\bar{A}$, we obtain its local representation. Let

$$v \in T_pM, \xi \in E_p, v = v^i \frac{\partial}{\partial x^i}(p), \xi = \xi^m \psi_m(p),$$

and Γ^n_{im} be the Christoffel symbols of ∇. Then

$$\bar{v}_\xi = v^i \frac{\partial}{\partial \bar{x}^i}(\xi) - v^i \xi^n \Gamma^m_{in}(p) \frac{\partial}{\partial \psi^m}(\xi)$$

[9]. So if A is expressed as $A(\xi) = A^i(\xi)\frac{\partial}{\partial x^i}$ $(A^i = dx^i \circ A)$ then

$$\bar{A}_\xi = A^i(\xi)\frac{\partial}{\partial \bar{x}^i}(\xi) - A^i(\xi)\xi^n \Gamma^m_{in}(p)\frac{\partial}{\partial \psi^m}(\xi)$$

For example if $E = TM$ and $A = 1_{TM}$, then $\overline{1_{TM}}$ is the geodesic spray of ∇. The set of all horizontal vector fields on E as well as the set of all strong bundle maps from E to TM are modules over $C^\infty(E)$.

From the definition of $\bar{A}$ and the local representations of A and $\bar{A}$ it is clear that the map $A \mapsto \bar{A}$ is a linear isomorphism between these modules.

For each $X \in \Gamma E$ (resp. $U \in \chi M$) $\widetilde{X \circ \pi}$ (resp. $\overline{U \circ \pi}$) is called the vertical lift of X (resp. the horizontal lift of U) and it is denoted by IX (resp. $\bar{U}$).

Proposition 1. Let $F : E \to E$ and $A : E \to TM$ be linear strong bundle maps, and R be the curvature tensor of ∇ then for X, $Y \in \Gamma E$ and U, $V \in \chi M$ we have

$$[IX, IY] = 0 \tag{1}$$

$$[\bar{U}, IX] = I\nabla_U X \tag{2}$$

$$[\bar{U}, \bar{V}] = \overline{[U, V]} - R(\widetilde{U, V}) \tag{3}$$

$$[IX, \tilde{F}] = IF \circ X \tag{4}$$

$$[\bar{U}, \tilde{F}] = \widetilde{\nabla_U F} \tag{5}$$

$$[IX, \bar{A}] = \overline{A \circ X} - \widetilde{\nabla_{A(.)}X} \tag{6}$$

$$[\bar{U}, \bar{A}] = \overline{L_U A} - R(U, \widetilde{\bar{A}(.)})(.) \tag{7}$$

In the above relations, $R(U, A(.))(.)$ denotes the bundle map

$$\xi \mapsto R(U \circ \pi(\xi), A(\xi))(\xi),$$

$A(.)$ denotes the bundle map $\xi \mapsto A(\xi)$ and $L_U A$: $E \to TM$ is the Lie derivative of A with respect to U given by

$$X \in \Gamma E \quad (L_U A)(X) = [U, A \circ X] - A(\nabla_U X)$$

Proof. For a proof of (1) and (2) see [9]. The relation (3) is followed by the definition of R (see [9]). To prove (5) and (7), note that for each $\xi \in E_p$ there exist some $X \in \Gamma E$ such that $X_p = \xi$ and for each $u \in T_pM$, $\nabla_u X = 0$ [9]. For such $X \in \Gamma E$, by a computation in local coordinates we obtain

$$[\bar{U}, \tilde{F}]_\xi = [\bar{U}, IF \circ X]_\xi$$

$$[\bar{U}, \bar{A}]_\xi = [\bar{U}, \overline{A \circ X}]_\xi$$

so

$$[\bar{U}, \tilde{F}]_\xi = [\bar{U}, IF \circ X]_\xi = (I\nabla_U F \circ X)_\xi = (I[(\nabla_U F)(X) + F(\nabla_U X)])_\xi = I_\xi(\nabla_U F)(X_p) + I_\xi F(\nabla_{U_p} X) = I_\xi(\nabla_U F)(\xi) = (\widetilde{\nabla_U F})_\xi$$

And for the relation (7) we have

$$[\bar{U}, \bar{A})_\xi = [\bar{U}, \overline{A \circ X}]_\xi = (\overline{[U, A \circ X]})_\xi - R(U, \widetilde{\bar{A}(.)})(.))_\xi = (\overline{(L_U A)(X) + A(\nabla_U X)})_\xi - I_\xi R(U_p, A(X_p))(\xi)$$

$$=\overline{((L_U A)(\xi))}_\xi - I_\xi R(U_p, A(\xi))(\xi) = \overline{(L_U A)}_\xi - (R(U, \widetilde{A}(.))(.))_\xi$$

The assertion (6) is also proved by a computation in a local coordinate. The equality (4) is proved by using the definition of Lie derivative and the flow of IX which is $\varphi_t(\xi)=\xi+tX_{\pi(\xi)}$.•

Lift of Semi-Riemannian Metrics

The linear bundle map h from the tangent bundle $TE \xrightarrow{\pi_E} E$ into the vector bundle $E \oplus TM \to M$, defined by $h: \hat{v} \mapsto (k(\hat{v}), d\pi(\hat{v}))$ is an isomorphism over each fiber. Thus $TE \xrightarrow{\pi_E} E$ is the pull-back of $E \oplus TM \to M$ over $\pi: E \to M$ and we can see that

$$X \in \Gamma E,\ U \in \chi M \quad h^{\#}(X+U) = IX + \bar{U}$$

(for definition of $h^{\#}$ see [3]). Vector fields of the form IX and $\bar{U}$ generate the $C^\infty(E)$- module χE [3].

Now, let E be a semi-Riemannian vector bundle, and M be a semi-Riemannian manifold. Then we can naturally make $E \otimes TM$ into a semi-Riemannian vector bundle and by using the map h, we can define a semi-Riemannian metric $<>$ on E as follows:

$$\hat{u}, \hat{v} \in T_\xi E \quad <\hat{u}, \hat{v}> = <k(\hat{u}), k(\hat{v})>_E + <d\pi(\hat{u}), d\pi(\hat{v})>_M$$

Thus E becomes a semi-Riemannian manifold. At each point $\xi \in E$, the horizontal space $\mathcal{H}_\xi$ and the vertical space $(vE)_\xi$ are orthogonal to each other, and the inner product on $\mathcal{H}_\xi$ and $(vE)_\xi$ are the same as the inner products on $T_{\pi(\xi)} M$ and $E_{\pi(\xi)}$ under the isomorphism $d\pi: \mathcal{H}_\xi \to T_{\pi(\xi)} M$ and $k: (vE)_\xi \to E_{\pi(\xi)}$, respectively. So scalar products of horizontal and vertical vector fields of E are zero.

Let C be a set and G be a semi-Riemannian vector bundle. Assume that $f, g: C \to G$ are functions such that for each $x \in C$, $f(x)$ and $g(x)$ are in the same fiber. Then $<f, g>$ denotes the function from C to IR given by

$$\forall x \in C \qquad <f,g>(x) = <f(x), g(x)>$$

Now, let $F_1, F_2: E \to E$ and $A_1, A_2: E \to TM$ be strong bundle maps (not necessarily linear) clearly

$$<\widetilde{F}_1, \widetilde{F}_2>_{TE} = <F_1, F_2>_E \quad <\overline{A}_1, \overline{A}_2>_{TE} = <A_1, A_2>_M$$

From now on we assume that the metric of the vector bundle is parallel with respect to ∇, namely for every $X, Y \in \Gamma E$ and $U \in \chi M$ we have

$$U<X,Y> = <\nabla_U X, Y> + <X, \nabla_U Y>$$

The Levi-Civita Connection of E

To compute the Levi-Civita connection of E, we need derivations of some functions on E along some fundamental vector fields.

Proposition 2. Let $X, Y \in \Gamma E$, $U, V \in \chi M$ and $F: E \to E$, $A: E \to TM$ be linear strong bundle maps. Then

$$\bar{V}<F, X\circ\pi> = <\nabla_V F, X\circ\pi> + <F, (\nabla_V X)\circ\pi> \quad (1)$$
$$IY<F, X\circ\pi> = <F\circ Y, X>\circ\pi \quad (2)$$
$$\bar{V}<A, U\circ\pi> = <\nabla_V A, U\circ\pi> + <A, (\nabla^M_V U)\circ\pi> \quad (3)$$
$$IY<A, U\circ\pi> = <A\circ Y, U>\circ\pi \quad (4)$$

Proof. The proof of (1) (resp. (2)) is the same as the proof of (3) (resp. (4)). So we prove (1) and (2). By definition

$$(IY)_\xi = I_\xi Y_{\pi(\xi)} = \frac{d}{dt}\Big|_{t=0} (\xi + tY_{\pi(\xi)})$$

so for $\pi(\xi) = p$

$$\begin{aligned}(IY<F, X\circ\pi>)_\xi &= (IY)_\xi <F, X\circ\pi> \\ &= \tfrac{d}{dt}\big|_{t=0} <F, X\circ\pi>(\xi + tY_p) \\ &= \tfrac{d}{dt}\big|_{t=0} <F(\xi + tY_p), X_p> \\ &= \tfrac{d}{dt}\big|_{t=0} (<F(\xi), X_p> + t<F(Y_p), X_p>) \\ &= <F(Y_p), X_p> = <F\circ Y, X>\circ\pi(\xi)\end{aligned}$$

This proves (2). Now if $V_p = 0$, then concerning (1) there is nothing to prove. So let $V_p \neq 0$, and suppose that $\alpha:]-\varepsilon, \varepsilon[\to M$ is a curve such that $\alpha'(0) = V_p$. Set $\bar{\alpha}(t) = (\mathrm{IP}_\alpha \xi)(t)$. So $\bar{\alpha}'(0) = (\bar{V})_\xi$. We can find a section of E say Y, such that for small t, $Y_{\alpha(t)} = \bar{\alpha}(t)$, so $\nabla_{V_p} Y = 0$. Now we are ready to prove (1). From the above we have

$$(\bar{V}<F,X\circ\pi>)(\xi)=\bar{V}_\xi<F,X\circ\pi>$$

$$=\frac{d}{dt}|_{t=0}<F,X\circ\pi>(\bar{\alpha}(t))$$

$$=\frac{d}{dt}|_{t=0}<F(\bar{\alpha}(t)),X_{\alpha(t)}>=\frac{d}{dt}|_{t=0}<F(Y_{\alpha(t)}),X_{\alpha(t)}>$$

$$=\frac{d}{dt}|_{t=0}<F\circ Y,X>(\alpha(t))=\alpha'(0)<F\circ Y,X>$$

$$=V_p<F\circ Y,X>$$

$$=<\nabla_{V_p}(F\circ Y),X_p>+<(F\circ Y)(p),\nabla_{V_p}X>$$

$$=<(\nabla_{V_p}F)(Y_p)+F(\nabla_{V_p}Y),X_p>+<F(Y_p),\nabla_{V_p}X>$$

$$=<(\nabla_{V_p}F)(\xi),X_p>+<F(\xi),\nabla_{V_p}X>$$

$$=<\nabla_V F,X\circ\pi>(\xi)+<F,(\nabla_V X)\circ\pi>(\xi).\bullet$$

Let $\diamondsuit(E)$ be the vector bundle over M, whose fiber at each point $p\in M$ is $\diamondsuit(E_p)$ and let $L(\wedge^2TM\ ,\ \diamondsuit(E))$ be the vector bundle over M, whose fiber at each point $p\in M$ is $L(\wedge^2T_pM\ ,\ \diamondsuit(E_p))$ (space of linear maps between these vector spaces). Then R(the curvature tensor of ∇) is a section of $L(\wedge^2TM\ ,\ \diamondsuit(E))$. As mentioned above, $\diamondsuit(E)$ and $\diamondsuit(TM)$ are naturally isomorphic to $\wedge^2E$ and $\wedge^2TM$. So we use them interchangeably, and assume that

$$R\in\Gamma\,\mathrm{L}(\wedge^2TM\ ,\ \wedge^2E).$$

Then

$$R^*\in\Gamma\,\mathrm{L}(\wedge^2E\ ,\ \wedge^2TM).$$

or

$$R^*\in\Gamma\,\mathrm{L}(\wedge^2E\ ,\ \diamondsuit(TM)).$$

which is defined explicitly and uniquely by the following formula

$$X\ ,\ Y\in\Gamma\,E\ ,\ U\ ,\ V\in\chi M\qquad <R(U\ ,\ V)(X),Y>_E$$

$$=<R^*(X\ ,\ Y)(U),V>_M.$$

For example if $E=TM$, and $\nabla=\nabla^M$ (the Levi-Civita connection of M), then $R^*=R$. In other words, R is symmetric with respect to the inner product of $\wedge^2TM$.

Theorem 3. Let $\bar{\nabla}$ denote the Levi-Civita connection of E. If $F:E\to E$ and $A:E\to TM$ are linear strong bundle maps and $X,Y\in\Gamma E$, $U,V\in\chi\ M$, then

$$\bar{\nabla}_{IX}IY=0 \qquad (5)$$

$$\bar{\nabla}_{\bar{U}}\bar{V}=\overline{\nabla^M_U V}-\frac{1}{2}R(\widetilde{U,V}) \qquad (6)$$

$$\bar{\nabla}_{IX}\bar{U}=\frac{1}{2}\overline{R^*(.,X)(U)} \qquad (7)$$

$$\bar{\nabla}_{IX}\tilde{F}=IF\circ X \qquad (8)$$

$$\bar{\nabla}_{IX}\bar{A}=\overline{A\circ X}+\frac{1}{2}\overline{R^*(.,X)(A(.))} \qquad (9)$$

$$\bar{\nabla}_{\bar{U}}\tilde{F}=\widetilde{\nabla_U F}+\frac{1}{2}\overline{R^*(.,F(.))(U)} \qquad (10)$$

$$\bar{\nabla}_{\bar{U}}\bar{A}=\overline{\nabla_U A}-\frac{1}{2}R(U,\widetilde{A(.)})(.) \qquad (11)$$

Proof. The theorem is a consequence of the identities in section 2 and this section and the following two assertions:

1) The Levi-Civita connection of a semi-Riemannian manifold N is uniquely determined by the following formula [9]. For $U,V,W\in\chi N$

$$2<\nabla_U V,W>=U<V,W>+V<W,U>-W<U,V>$$
$$+<[U,V],W>-<[V,W],U>+<[W,U],V>$$

2) Two vector fields $\hat{U},\hat{V}\in\chi\ E$ are equal if and only if for each $Z\in\Gamma\,E$ and $W\in\chi\ M$

$$<\hat{U},\bar{W}>=<\hat{V},\bar{W}>,<\hat{U},IZ>=<\hat{V},IZ>$$

We prove only the assertions (10) and (11). Let us prove (10).

$$2<\bar{\nabla}_{\bar{U}}\tilde{F},IZ>=\bar{U}<\tilde{F},IZ>+\tilde{F}<IZ,\bar{U}>-IZ<\bar{U},\tilde{F}>$$
$$+<[\bar{U},\tilde{F}],IZ>-<[\tilde{F},IZ],\bar{U}>+<[IZ,\bar{U}],\tilde{F}>$$
$$=\bar{U}<F,Z\circ\pi>+<\widetilde{\nabla_U F},IZ>-<-I(F\circ Z),$$

$\bar{U}> + <-I\nabla_U Z, \tilde{F}>$

$= <\nabla_U F, Z\circ\pi> + <F, (\nabla_U Z)\circ\pi> + <\nabla_U F,$

$Z\circ\pi> - <(\nabla_U Z)\circ\pi, F>$

$= 2<\nabla_U F, Z\circ\pi> = 2<\overline{\nabla_U F + \frac{1}{2}R^*(.,F(.))(U)}, IZ>$

$2<\bar{\nabla}_{\bar{U}}\tilde{F}, \bar{W}> = \bar{U}<\tilde{F}, \bar{W}> + \tilde{F}<\bar{W}, \bar{U}> - \bar{W}<\bar{U}, \tilde{F}>$

$+<[\bar{U}, \tilde{F}], \bar{W}> - <[\tilde{F}, \bar{W}], \bar{U}> + <[\bar{W}, \bar{U}], \tilde{F}>$

$= \tilde{F}<W, U>\circ\pi + <\widetilde{\nabla_u F}, \bar{W}> - <-\widetilde{\nabla_W F}, \bar{U}>$

$+<\overline{[W, U]} - R(\widetilde{W}, U), \tilde{F}>$

$= d\pi(\tilde{F})<W, U> - <R(\widetilde{W}, U), \tilde{F}>$

$= -<R(W, U), F> = <R(U, W)(.), F(.)>$

$= <R^*(., F(.))(U), W> = 2<\widetilde{\nabla_U F} + \overline{\tfrac{1}{2}R^*(., F(.))(U)}, \bar{W}>$

Now we prove (11).

$<\bar{\nabla}_{\bar{U}}\bar{A}, IZ> = \bar{U}<\bar{A}, IZ> + \bar{A}<IZ, \bar{U}> - IZ<\bar{U}, \bar{A}>$

$+<[\bar{U}, \bar{A}], IZ> - <[\bar{A}, IZ], \bar{U}> + <[IZ, \bar{U}], \bar{A}>$

$= -IZ<U\circ\pi, A> + <\overline{L_U A} - R(U, \widetilde{A(.)})(.),$

$IZ> - <-\overline{A\circ Z} + \overline{\nabla_{A(.)}}Z, \bar{U}>$

$+<-I\nabla_U Z, \bar{A}> = -<U, A\circ Z> - <R(U, \widetilde{A(.)})(.), IZ>$

$+<A\circ Z, U> = 2<\overline{\nabla_U A} - \tfrac{1}{2}R(U, \widetilde{A(.)})(.), IZ>$

To complete the proof, fix $\xi \in E_p$ and suppose X is a section of E such that $X_p = \xi$ and for each $u \in T_pM$, $\nabla_u X = 0$. As mentioned before for each $U\ \xi\ \mathfrak{X}M$ we have $[\bar{U}, \bar{A}]_\xi = [\bar{U}, \overline{A\circ X}]_\xi$. By a direct computation we can see that for each $U, V \in \mathfrak{X}M$

$$\bar{U}_\xi<A, V\circ\pi> = U_p<A\circ X, V>.$$

Now

$2<\bar{\nabla}_{\bar{U}}\bar{A}, \bar{W}>(\xi) = \bar{U}_\xi<\bar{A}, \bar{W}> + \bar{A}_\xi<\bar{W}, \bar{U}> - \bar{W}_\xi<\bar{U}, \bar{A}>$

$+<[\bar{U}, \bar{A}], \bar{W}>(\xi) - <[\bar{A}, \bar{W}], \bar{U}>(\xi) + <[\bar{W}, \bar{U}], \bar{A}>(\xi)$

$= \bar{U}_\xi<A, W\circ\pi> + \bar{A}_\xi<W, U>\circ\pi - \bar{W}_\xi<U\circ\pi, A>$

$+<[\bar{U}, \overline{A\circ X}], \bar{W}>(\xi) - <[\overline{A\circ X}, \bar{W}], \bar{U}>(\xi)$

$+<[\bar{W}, \bar{U}], \bar{A}>(\xi)$

$= U_p<A\circ X, W> + (A\circ X)_p<W, U> - W_p<U, A\circ X>$

$+<[U, A\circ X], W>(p) - <[A\circ X, W],$

$U>(p) + <[W, U], A\circ X>(p)$

$= 2<\nabla^M_U(A\circ X), W>(p) = 2<(\nabla_U A)(X) + A(\nabla_U X), W>(p)$

$= 2<\overline{\nabla_U A}, \bar{W}>(\xi) = 2<\overline{\nabla_U A - \tfrac{1}{2}R(U, \widetilde{A(.)})(.)}, \bar{W}>(\xi)$.

Geodesics of E

We know that for each $\xi \in E_p$ and $\hat{v} \in T_\xi E$ there exists a unique geodesic $\gamma:]-\varepsilon, \varepsilon[\rightarrow E$ such that $\gamma(0) = \xi$ and $\gamma'(0) = \hat{v}$. To determine it, first suppose that $\hat{v}$ is vertical. So $\hat{v} = I_\xi\eta$, for some $\eta \in E_p$. Let γ be the curve defined by $\gamma(t) = \xi + t\eta$ which is entirely in the fiber E_p. Trivially, $\gamma(0) = \xi$ and $\gamma'(0) = I_\xi\eta = \hat{v}$

Now choose $X \in \Gamma E$ in such a way that $X_p = \eta$, so $\gamma'(t) = (IX)_{\gamma(t)}$, thus

$$\bar{\nabla}_{\gamma(t)}\gamma' = (\bar{\nabla}_{IX}IX)_{\gamma(t)} = 0$$

Therefore, γ is the desired geodesic.

Clearly, the fibers of E are semi-Riemannian submanifolds of E and the restriction of the metric of E to them is their orginal metrics. By the above result, they are geodesically complete.

Now suppose that $\hat{v}$ is not vertical and γ is the desired geodesic. Thus $\gamma'(t)$ is never vertical. Set $\alpha = \pi \circ \gamma$. α is a curve in M, and $\alpha'(t) = d\pi(\gamma'(t)) \neq 0$. Consequently we can choose $U \in \mathfrak{X}M$ such that for small t, $U_{\alpha(t)} = \alpha'(t)$. We can also choose $X \in \Gamma E$ in such a way that for small t, $X_{\alpha(t)} = \gamma(t)$. To compute $\bar{\nabla}_\gamma\gamma'$, we find a suitable vector field on E for which γ is an integral curve for small values of t. Clearly we have

$$d\pi(\gamma'(t)) = \alpha'(t) = U_{\alpha(t)}$$

and

$$k(\gamma'(t)) = k((X \circ \alpha)'(t)) = k(dX(\alpha'(t)))$$

$$= \nabla_{\alpha'(t)} X = (\nabla_U X)_{\alpha(t)}$$

Therefore, if we set $Y = \nabla_U X$, then $\bar{U} + IY$ is the suitable vector field.

Set, $v = d\pi(\hat{v}), \eta = k(\hat{v})$. So $\alpha'(0) = U_p = v$, $Y_p = \eta$, $\alpha(0) = p$. Since γ is a geodesic we have

$$0 = \bar{\nabla}_{\gamma'(t)} \gamma' = (\bar{\nabla}_{\bar{U}+IY} (\bar{U} + IY))(\gamma(t))$$

$$= (\bar{\nabla}_{\bar{U}} \bar{U} + \bar{\nabla}_{\bar{U}} IY + \bar{\nabla}_{IY} \bar{U} + \bar{\nabla}_{IY} IY)(\gamma(t))$$

$$= (\overline{\nabla^M_U U} - \tfrac{1}{2} R(\widetilde{U}, U) + \overline{R^*(., Y)(U)} + I\nabla_U Y)(\gamma(t))$$

The above relations imply the following

$$(\nabla^M_U U)(\alpha(t)) + R^*(\gamma(t), Y_{\alpha(t)})(U_{\alpha(t)}) = 0 \quad (1)$$
$$(\nabla_U Y)(\alpha(t)) = 0 \quad (2)$$

Now, recall that for a differentiable curve $h : I \to E_p$, and $\alpha : I \to M$ with $\alpha(0) = p$, we can define a section of E along α as follows:

$$t \in I \qquad X(t) = (\mathrm{IP}_\alpha h(t))(t)$$

The covariant derivative of X, along α' is as follows:

$$\nabla_{\alpha'(t)} X = (\mathrm{IP}_\alpha h'(t))(t)$$

By repeating this formula for second derivative we have

$$\nabla_{\alpha'(t)} \nabla_{\alpha'} X = (\mathrm{IP}_\alpha h''(t))(t)$$

In the special case of our problem, we can assume that $X_{\alpha(t)} = (\mathrm{IP}_\alpha h(t))(t)$, for some $h : I \to E_p$

$$(\mathrm{IP}_\alpha h''(t))(t) = \nabla_{\alpha'(t)}(\nabla_{\alpha'} X) = \nabla_{\alpha'(t)} Y = (\nabla_U Y)(\alpha(t)) = 0$$

Thus, $h'' = 0$. But

$$h(0) = (\mathrm{IP}_\alpha h(0))(0) = X_{\alpha(0)} = \gamma(0) = \xi$$

and

$$h'(0) = (\mathrm{IP}_\alpha h'(0))(0) = \nabla_{\alpha'(0)} X = Y_{\alpha(0)} = Y_p = \eta.$$

So $h(t) = \xi + t\eta$. Therefore, the geodesic γ is of the following form

$$\gamma(t) = (\mathrm{IP}_\alpha(\xi + t\eta))(t)$$

and α is a curve in M for which we have

$$\nabla_{\alpha'(t)} \alpha' + R^*(\gamma(t), Y_{\alpha(t)})(\alpha'(t)) = 0$$

From this equality we have

$$\nabla_{\alpha'(t)} \alpha' + R^*((\mathrm{IP}_\alpha(\xi + t\eta))(t), (\mathrm{IP}_\alpha \eta)(t))(\alpha'(t)) = 0$$

or

$$\nabla_{\alpha'(t)} \alpha' + R^*((\mathrm{IP}_\alpha \xi)(t), (\mathrm{IP}_\alpha \eta)(t))(\alpha'(t)) = 0 \quad (3)$$

So, α is a solution of the equation (3), with initial condition, $\alpha'(0) = v$. In some special cases, the solutions of (3) can easily be found. For example, if ∇ is flat ($R = 0$), then the solutions of (3) are geodesics of M. If $\hat{v}$ is horizontal ($\eta = 0$), then α is a geodesic of M. Thus, geodesics of E with horizontal tangent vector (at some point), are obtained by parallel transport of some points of E along some geodesics of M.

Curvature Tensor of E

Theorem 4. Let $\bar{R}$ and R^M denote the curvature tensors of $\bar{\nabla}$ and ∇^M, respectively. Assume that $X, Y, Z \in \Gamma E$ and $U, V, W \in \mathfrak{X} M$. Then

$$\bar{R}(IX, IY)(IZ) = 0 \quad (1)$$

$$\bar{R}(\bar{U}, IX)(IY) = -\frac{1}{2}[\overline{R^*(X, Y)(U)} + \frac{1}{2}\overline{R^*(., X)(R^*(., Y))(U)}] \quad (2)$$

$$\bar{R}(IX, IY)(\bar{U}) = \overline{R^*(X, Y)(U)} + \tfrac{1}{4}[\overline{R^*(., X)(R^*(., Y)(U))} - \overline{R^*(., Y)(R^*(., X)(U))}] \quad (3)$$

$$\bar{R}(\bar{U}, IX)(\bar{V}) = \tfrac{1}{2} I(R(U, V)(X)) - \tfrac{1}{4} R(U, R^*(\widetilde{., X})(V))(.) + \tfrac{1}{2}\overline{(\nabla_U R^*)(., X)(V)} \quad (4)$$

$$\bar{R}(\bar{U}, \bar{V})(IX) = I(R(U, V)(X)) - \tfrac{1}{2}\overline{(\nabla_V R^*)(., X)(U)} + \tfrac{1}{2}\overline{(\nabla_U R^*)(., X)(V)} - \tfrac{1}{4} R(U, R^*(\widetilde{., X})(V))(.)$$

$$+\frac{1}{4}R(V,\widetilde{R^*(.,X)(U)})(.) \qquad (5)$$

$$\bar{R}(\bar{U},\bar{V})(\bar{W})=\overline{R^M(U,V)(W)}+\frac{1}{2}\widetilde{(\nabla_W R)}(U,V)$$

$$+\frac{1}{2}\overline{R^*(.,R(U,V)(.))(W)}-\frac{1}{4}\overline{R^*(.,R(V,W)(.))(U)}$$

$$+\frac{1}{4}\overline{R^*(.,R(U,W)(.))(V)} \qquad (6)$$

Proof. The proof is by direct computation and all these computations are short except (6) which is lengthy. We compute the relation (4).

$$\bar{R}(\bar{U},IX)(\bar{V})=\bar{\nabla}_{\bar{U}}\bar{\nabla}_{IX}\bar{V}-\bar{\nabla}_{IX}\bar{\nabla}_{\bar{U}}\bar{V}-\bar{\nabla}_{[\bar{U},IX]}\bar{V}$$

$$=\bar{\nabla}_{\bar{U}}(\frac{1}{2}\overline{R^*(.,X)(V)})-\bar{\nabla}_{IX}(\overline{\nabla_U V}-\frac{1}{2}\widetilde{R}(\bar{U},V))-\bar{\nabla}_{I\nabla_U X}\bar{V}$$

$$=\frac{1}{2}\overline{\nabla_U(R^*(.,X)(V))}-\frac{1}{4}R(U,\widetilde{R^*(.,X)(V)})(.)$$

$$-\frac{1}{2}\overline{R^*(.,X)(\nabla_U V)}+\frac{1}{2}I(R(U,V)(X))$$

$$-\frac{1}{2}\overline{R^*(.,\nabla_U X)(V)}=\frac{1}{2}I(R(U,V)(X))$$

$$-\frac{1}{4}R(U,\widetilde{R^*(.,X)(V)})(.)+\frac{1}{2}\overline{(\nabla_U R^*)(.,X)(V)}\cdot$$

Proposition 5. Let the metric of M be positive or negative definite. Then $\bar{\nabla}\bar{R}=0$ if and only if $R=0$ and $\nabla R^M=0$.

Proof. First suppose $R=0$ and $\nabla R^M=0$ (this part of the proposition needs no special assumption on the metric of M). By using formulas (1)-(6) and merely the assumption $R=0$, we find that when at least one of the vector fields $\hat{X},\hat{Y}$ and $\hat{Z}$ is vertical then $\bar{R}(\hat{X},\hat{Y})(\hat{Z})$ is zero. Now let $U,V,W\in\mathfrak{X}M$. Then

$$\bar{R}(\bar{U},\bar{V})(\bar{W})=\overline{R^M(U,V)(W)}.$$

To prove that $\bar{\nabla}\bar{R}=0$, we check all possible cases. For example

$$(\bar{\nabla}_{IX}\bar{R})(IY_1,IY_2)(IY_3)=\bar{\nabla}_{IX}(\bar{R}(IY_1,IY_2)(IY_3))$$

$$-\bar{R}(\bar{\nabla}_{IX}IY_1,IY_2)(IY_3)-\bar{R}(IY_1,\bar{\nabla}_{IX}IY_2)(IY_3)$$

$$-\bar{R}(IY_1,IY_2)(\bar{\nabla}_{IX}IY_3)=0$$

The proof of the assertion in other cases is in the same way. In computation, there appear expressions which involve values of R or R^* which are zero. The only case for which we use $\nabla R^M=0$ is the following

$$(\bar{\nabla}_{\bar{U}}\bar{R})(\bar{V}_1,\bar{V}_2)(\bar{V}_3)=\bar{\nabla}_{\bar{U}}(\bar{R}(\bar{V}_1,\bar{V}_2)(\bar{V}_3))-\bar{R}(\bar{\nabla}_{\bar{U}}\bar{V}_1,\bar{V}_2)(\bar{V}_3)$$

$$-\bar{R}(\bar{V}_1,\bar{\nabla}_{\bar{U}}\bar{V}_2)(\bar{V}_3)-\bar{R}(\bar{V}_1,\bar{V}_2)(\bar{\nabla}_{\bar{U}}\bar{V}_3)=\bar{\nabla}_{\bar{U}}\overline{R^M(V_1,V_2)(V_3)}$$

$$-\bar{R}(\overline{\nabla_U V_1},\bar{V}_2)(\bar{V}_3)-\bar{R}(\bar{V}_1,\overline{\nabla_U V_2})(\bar{V}_3)-\bar{R}(\bar{V}_1,\bar{V}_2)(\overline{\nabla_U V_3})=$$

$$\overline{\nabla_U R^M(V_1,V_2)(V_3)}-\overline{R^M(\nabla_U V_1,V_2)(V_3)}-\overline{R^M(V_1,\nabla_U V_2)(V_3)}$$

$$-\overline{R^M(V_1,V_2)(\nabla_U V_3)}=\overline{(\nabla_U R^M)(V_1,V_2)(V_3)}=0$$

Now assume that $\bar{\nabla}\bar{R}=0$. Let $X,Y,Z\in\Gamma E$ and $U\in\mathfrak{X}M$. Then

$$0=(\bar{\nabla}_{\bar{U}}\bar{R})(IX,IY)(IZ)=\bar{\nabla}_{\bar{U}}(\bar{R}(IX,IY)(IZ))$$

$$-\bar{R}(\bar{\nabla}_{\bar{U}}IX,IY)(IZ)-\bar{R}(IX,\bar{\nabla}_{\bar{U}}IY)(IZ)-\bar{R}(IX,IY)$$

$$(\bar{\nabla}_{\bar{U}}IZ)=\bar{R}(IY,\bar{\nabla}_{\bar{U}}IX)(IZ)-\bar{R}(IX,\bar{\nabla}_{\bar{U}}IY)(IZ)$$

$$-\bar{R}(IX,\bar{\nabla}_{\bar{U}}IZ)(IY)+\bar{R}(IY,\bar{\nabla}_{\bar{U}}IZ)(IX)$$

All the terms in the above expression are in the same form. We compute one of them.

$$\bar{R}(IY,\bar{\nabla}_{\bar{U}}IX)(IZ)=\bar{R}(IY,I\nabla_U X+\frac{1}{2}\overline{R^*(.,X)(U)})(IZ)$$

$$=\bar{R}(IY,\frac{1}{2}\overline{R^*(.,X)(U)})(IZ)=\frac{1}{4}[\overline{R^*(Y,Z)(R^*(.,X)(U))}$$

$$+\frac{1}{2}\overline{R^*(.,Y)(R^*(.,Z)(R^*(.,X)(U)))}]$$

So by substitution, we obtain

$$0=R^*(Y,Z)\circ R^*(.,X)(U)+\frac{1}{2}R^*(.,Y)\circ R^*(.,Z)$$

$$\circ R^*(.,X)(U)-R^*(X,Z)\circ R^*(.,Y)(U)-\frac{1}{2}R^*(.,X)$$

$$\circ R^*(.,Z)\circ R^*(.,Y)(U)-R^*(X,Y)\circ R^*(.,Z)(U)$$

$$-\frac{1}{2}R^*(.,X)\circ R^*(.,Y)\circ R^*(.,Z)(U)+R^*(Y,X)$$

$$\circ R^*(.,Z)(U)+\frac{1}{2}R^*(.,Y)\circ R^*(.,X)\circ R^*(.,Z)(U)$$

In putting X and Z in turn in place of the dot in the above expression we obtain

$$-R^*(X,Z)\circ R^*(X,Y)(U)-2R^*(X,Y)\circ R^*(X,Z)(U)=0 \quad (7)$$

$$R^*(Y,Z)\circ R^*(Z,X)(U)-R^*(X,Z)\circ R^*(Z,Y)(U)=0 \quad (8)$$

Now interchange X and Z in (8) and sum the resulting expression with (7). From this and the fact that U is arbitrary, we obtain

$$R^*(X,Y)\circ R^*(X,Z)=0$$

Now, set $Y = Z$, and $T = R^*(X, Y)$. Clearly, T is an antisymmetric map for which $T^2 = 0$. But such an antisymmetric (or symmetric) map is zero, because

$$0=<T^2(v), v>=\pm<T(v), T(v)>=\pm\|T(v)\|^2 \Rightarrow T(v)=0$$

Consequently, $R^* = 0$ and so $R = 0$. Now by the computation in the first part of the proposition we have

$$0=(\bar{\nabla}_{\bar{U}}\bar{R})(\bar{V}_1,\bar{V}_2)(\bar{V}_3)=\overline{(\nabla_U R^M)(V_1,V_2)(V_3)} \Rightarrow \nabla R^M=0 \bullet$$

Sectional, Ricci, and Scalar Curvatures of E

With $\bar{R}$ in hand, we can easily compute other curvatures of E. In general, if σ is a non-degenerate plane of $T_\xi E$, for some $\xi \in E$, and $\{\hat{\upsilon},\hat{u}\}$ is a base of σ, then the sectional curvature of E along σ, denoted by $\bar{K}_\sigma$ is

$$\bar{K}_\sigma=\frac{<\bar{R}(\hat{u},\hat{\upsilon})(\hat{\upsilon}),\hat{u}>}{<\hat{u},\hat{u}><\hat{\upsilon},\hat{\upsilon}>-<\hat{u},\hat{\upsilon}>^2}$$

(The denominator is non-zero if and only if σ is non-degenerate). To indicate that $\{\hat{u},\hat{\upsilon}\}$ is a base of σ, we denote σ by $\sigma(\hat{u},\hat{\upsilon})$.

Proposition 6. Let $p \in M$, u, $\upsilon \in T_pM$ and ξ, ζ, $\eta \in E_p$. Assume that

$$<u,\upsilon>=0, <\eta,\zeta>=0, |<u,u>|=|<\upsilon,\upsilon>|$$
$$=|<\eta,\eta>|=|<\zeta,\zeta>|=1.$$

Let K be the sectional curvature of M. Then

$$\bar{K}_{\sigma(\bar{u}_\xi,\bar{\upsilon}_\xi)}=K_{\sigma(u,\upsilon)}-\frac{3}{4}sgn(u)\,sgn(\upsilon)\|R(u,\upsilon)(\xi)\|^2 \quad (1)$$

$$\bar{K}_{\sigma(\bar{u}_\xi,I_\xi\eta)}=\frac{1}{4}sgn(u)\,sgn(\eta)\|R^*(\xi,\eta)(u)\|^2 \quad (2)$$

$$\bar{K}_{\sigma(I_\xi\eta,I_\xi\zeta)}=0 \quad (3)$$

Proof. Clearly

$$<\bar{u}_\xi,\bar{u}_\xi><\bar{\upsilon}_\xi,\bar{\upsilon}_\xi>-<\bar{u}_\xi,\bar{\upsilon}_\xi>^2=sgn(u)\,sgn(\upsilon)$$

and

$$<\bar{R}(\bar{u}_\xi,\bar{\upsilon}_\xi)(\bar{\upsilon}_\xi),\bar{u}_\xi>=$$
$$<\overline{(R^M(u,\upsilon)(\upsilon))}_\xi+\tfrac{1}{2}I_\xi(\nabla_\upsilon R)(u,\upsilon)(\xi)$$
$$+\tfrac{1}{2}\overline{(R^*(\xi,R(u,\upsilon)(\xi))(\upsilon))}_\xi+\tfrac{1}{4}\overline{(R^*(\xi,R(\upsilon,\upsilon)(\xi))(u))}_\xi$$
$$+\tfrac{1}{4}\overline{(R^*(\xi,R(u,\upsilon)(\xi))(\upsilon))}_\xi,\bar{u}_\xi>=<R^M(u,\upsilon)(\upsilon),u>$$
$$+\tfrac{3}{4}R^*(\xi,R(u,\upsilon)(\xi))(\upsilon),u>=<R^M(u,\upsilon)(\upsilon),u>$$
$$+\tfrac{3}{4}<R(\upsilon,u)(\xi),R(u,\upsilon)(\xi)>=<R^M(u,\upsilon)(\upsilon),u>$$
$$-\tfrac{3}{4}\|R(u,\upsilon)(\xi)\|^2$$

So (1) is proved. On the other hand, clearly

$$<\bar{u}_\xi,\bar{u}_\xi><I_\xi\eta,I_\xi\eta>-<\bar{u}_\xi,I_\xi\eta>^2=sgn(u)\,sgn(\eta)$$

Furthermore

$$<\bar{R}(\bar{u}_\xi,I_\xi\eta)(I_\xi\eta),\bar{u}_\xi>=<-\tfrac{1}{2}\overline{(R^*(\eta,\eta)(u))}_\xi$$
$$-\tfrac{1}{4}\overline{(R^*(\xi,\eta)(R^*(\xi,\eta)(u)))}_\xi,\bar{u}_\xi>=$$
$$-\tfrac{1}{4}<R^*(\xi,\eta)(R^*(\xi,\eta)(u)),u>$$
$$=\tfrac{1}{4}<R^*(\xi,\eta)(u),R^*(\xi,\eta)(u)>=\tfrac{1}{4}\|R^*(\xi,\eta)(u)\|^2$$

So (2) is proved. In the same way (3) can be proved. •

Let $\xi \in E$. Assume that $\{\hat{e}_1, ..., \hat{e}_k\}$ is an orthonormal basis for $T_\xi E$. Let $\bar{S}$ (resp. $\bar{r}$) denote the Ricci curvature (resp. scalar curvature) of E. Then

$$\bar{S}(\hat{u}, \hat{v}) = \sum_{i=1}^{k} sgn(\hat{e}_i) < \bar{R}(\hat{e}_i, \hat{u})(\hat{v}), \hat{e}_i > \quad \hat{u}, \hat{v} \in T_\xi E$$

$$\bar{r}_\xi = \sum_{i=1}^{k} sgn(\hat{e}_i)\, \bar{S}(\hat{e}_i, \hat{e}_i)$$

Proposition 7. Let $p \in M$, $\xi \in E_p$. Assume that $\{v_1, ..., v_n\}$ and $\{\eta_1, ..., \eta_m\}$ are orthonormal bases for T_pM and E_p, respectively. If S (resp. r) denotes Ricci curvature (resp. scalar curvature) of M, then for $\zeta, \eta \in E_p$ and $u, v \in T_pM$ we have

$$\bar{S}(I_\xi\eta, I_\xi\zeta) = \frac{1}{4}\sum_{i=1}^{n} sgn(v_i) < R^*(\xi, \zeta)(v_i), R^*(\xi, \eta)(v_i) > \quad (4)$$

$$\bar{S}(\bar{u}_\xi, I_\xi\eta) = \frac{1}{2}\sum_{i=1}^{n} sgn(v_i) < (\nabla_{v_i} R^*)(\xi, \eta)(u), v_i > \quad (5)$$

$$\bar{S}(\bar{u}_\xi, \bar{v}_\xi) = S(u, v) + \frac{1}{4}\sum_{j=1}^{m} sgn(\eta_j) < R^*(\xi, \eta_j)(u), R^*(\xi, \eta_j)(v) > - \frac{3}{4}\sum_{i=1}^{n} sgn(v_i) < R(v_i, u)(\xi), R(v_i, v)(\xi) > \quad (6)$$

$$\bar{r}_\xi = r_p + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} sgn(v_i)\, sgn(\eta_j) \| R^*(\xi, \eta_j)(v_i)\|^2 - \frac{3}{4}\sum_{i,j=1}^{n} sgn(v_i)\, sgn(v_j) \| R(v_i, v_j)(\xi)\|^2 \quad (7)$$

The proof is by direct computation.

Proposition 8. If the metric of M is definite (positive or negative), then E is Ricci-flat if and only if $R = 0$ and M is Ricci flat.

Proof. Suppose $R = 0$ and M is Ricci flat. (This part of the proposition needs no special assumption on the metric of M). Relations (4)-(6) show that E is Ricci flat. Conversely, suppose E is Ricci flat. Let $p \in M$ and $\xi, \eta \in E_p$. Assume that $\{v_1, ..., v_n\}$ is an orthonormal basis for T_pM. Then

$$0 = \bar{S}(I_\xi\eta, I_\xi\eta) = \frac{1}{4}\sum_{i=1}^{n} \pm R^*(\xi, \eta)(v_i), R^*(\xi, \eta)(v_i) >$$

$$= \pm\frac{1}{4}\sum_{i=1}^{n} \| R^*(\xi, \eta)(v_i)\|^2$$

$$\Rightarrow \quad R^*(\xi, \eta)(v_i) = 0$$

Since, ξ and η are arbitrary and $\{v_1, ..., v_n\}$ is a basis for T_pM we have $R^* = 0$. Therefore $R = 0$. Now by relation (6), for arbitrary $u, v \in T_pM$ we have

$$0 = \bar{S}(\bar{u}_\xi, \bar{v}_\xi) = S(u, v) + 0$$

Therefore $S = 0$

Proposition 9. If the metric of E is an Einstein metric, then E is Ricci flat.

Proof. By the definition of Einstein metric there exist $\lambda \in \mathbb{R}$ such that for every $\hat{U}, \hat{V} \in \mathfrak{X}E$,

$$\bar{S}(\hat{U}, \hat{V}) = \lambda < \hat{U}, \hat{V} >$$

Thus, specially for each $\xi \in E$

$$\bar{S}(I_\xi\xi, I_\xi\xi) = \lambda < I_\xi\xi, I_\xi\xi > = \lambda < \xi, \xi >.$$

On the other hand we have

$$\bar{S}(I_\xi\xi, I_\xi\xi) = \frac{1}{4}\sum_{i=1}^{n} sgn(v_i) < R^*(\xi, \xi)(v_i), R^*(\xi, \xi)(v_i) > = 0.$$

So, for each $\xi \in E$, $\lambda < \xi, \xi > = 0$. But for some ξ we have $< \xi, \xi > \neq 0$, so $\lambda = 0$. Therefore E is Ricci flat. •

Sphere Bundles and Pseudo-Sphere Bundles

Let $(V, < >)$ be an inner product space. The set

$$\{ v \in V \mid < v, v > = 1 \}$$

is denoted by S and is called the sphere in V. Similarly the set

$$\{ v \in V \mid < v, v > = -1 \}$$

will be denoted by $\hat{S}$ and we shall call it the pseudo-sphere in V. When S (resp. $\hat{S}$) is not empty, it is a non-degenerate submanifold of V whose dimension is one less than that of V, and its sectional curvature is 1 (resp. -1). Let E be a semi-Riemannian vector bundle, its associated sphere bundle E_S and its pseudo-sphere bundle $E_{\hat{S}}$ are defined as follows:

$$E_S = \{ \xi \in E \mid < \xi, \xi > = 1 \}, \quad E_{\hat{S}} = \{ \xi \in E \mid < \xi, \xi > = -1 \}$$

The bundle E_S (resp. $E_{\hat{S}}$), if not empty, is a submanifold of E whose dimension is one less than that of E. Let Z denote the radial vector field of E ($Z = \widetilde{1_E}$). For each $\xi \in E_S$, (resp. $\xi \in E_{\hat{S}}$), Z_ξ is a non-degenerate vector, and by an elementary computation we see that $T_\xi E_S$ (resp. $T_\xi E_{\hat{S}}$) is equal to $Z_\xi^\perp$. So E_S (resp. $E_{\hat{S}}$) is a non-degenerate submanifold of E, and Z is orthogonal to it. Since Z is a vertical vector field for each $\xi \in E_S$ (resp. $\xi \in E_{\hat{S}}$) we have

$$\mathcal{H}_\xi \subseteq T_\xi E_S (\text{resp. } T_\xi E_{\hat{S}})$$

So, each horizontal vector field along E_S (resp. $E_{\hat{S}}$) is tangent to it.

Proposition 10. Let T be the Weingarten map of E_S (resp. $E_{\hat{S}}$) with respect to Z. For each $\hat{u} \in T_\xi E_S$ (resp. $T_\xi E_{\hat{S}}$)

$$T(\hat{u}) = -I_\xi k(\hat{u})$$

Proof. Let $d\pi(\hat{u}) = u$ and $k(\hat{u}) = \eta$, so $\hat{u} = \bar{u}_\xi + I_\xi \eta$. Now by definition of T

$$T(\hat{u}) = -\bar{\nabla}_{\hat{u}} Z = -\bar{\nabla}_{\bar{u}_\xi + Z_\xi \eta} \widetilde{1_E} = -\bar{\nabla}_{\bar{u}_\xi} \widetilde{1_E} - \bar{\nabla}_{I_\xi \eta} \widetilde{1_E}$$

$$= -I_\xi(\nabla_u 1_E)(\xi) - \frac{1}{2}\overline{(R^*(\xi, \xi)(u))_\xi} - I_\xi 1_E(\eta) = -I_\xi \eta = -I_\xi k(\hat{u}) \bullet$$

Proposition 11. If π and $\hat{\pi}$ denote second fundamental forms of E_S and $E_{\hat{S}}$ respectively, then

$$\hat{U}, \hat{V} \in \mathfrak{X}\, E_S \quad \pi(\hat{U}, \hat{V}) = -<k(\hat{U}), k(\hat{V})> Z$$

$$\hat{U}, \hat{V} \in \mathfrak{X}\, E_{\hat{S}} \quad \hat{\pi}(\hat{U}, \hat{V}) = <k(\hat{U}), k(\hat{V})> Z$$

Proof. As mentioned in section 1, for $\hat{U}, \hat{V} \in \mathfrak{X}\, E_S$, and $\xi \in E$ we have

$$\pi(\hat{U}_\xi, \hat{V}_\xi) = <T(\hat{U}_\xi), \hat{V}_\xi> sgn(Z_\xi) Z_\xi$$

$$= <-I_\xi k(\hat{U}_\xi), \hat{V}_\xi> Z_\xi = <k(\hat{U}_\xi), k(\hat{V}_\xi)> Z_\xi$$

For $E_{\hat{S}}$ a similar computation can be done. •

Now, we can easily compute the curvatures of E_S or $E_{\hat{S}}$. For example, we see that sectional curvatures of E_S and $E_{\hat{S}}$ in the direction of the planes which have at least one horizontal vector are the same as those of E, and in the direction of the vertical planes they are constantly + 1 and - 1, respectively.

References

1. Cordero, L. A. and de Leon, M. On the curvature of the induced Riemannian metric on the frame bundle of a Riemannian manifold. *J. Math. Pures Appl.*, **65**, 81-91, (1986).
2. Dombrowski, P. On the geometry of the tangent bundles. *J. Reine Angew. Math.*, **210**, (1962).
3. Greub, W., Halperin, S. and Vanstone, R. *Connection, curvature, and cohomology*. Academic Press, New York, San Francisco, London, (1976).
4. Hoffman, K. *Linear algebra*. Prentice-Hall, Inc. Englewood Cliffs, New Jersey, (1961).
5. Kobayashi, S. and Nomizu, K. *Foundation of differential geometry*. Vol. I. Interscience Publ., New York, (1963).
6. Kowalski, O. Curvature of the induced Riemannian metric on the tangent bundle of a Riemannian manifold. *J. Riene Angew Math.*, **250**, (1971).
7. Mok, K. P. On the differential geometry of frame bundles of Riemannian manifolds. *Ibid.*, **302**, (1978).
8. O'Neill, *Semi-Riemannian geometry with application to relativity*. Academic Press, (1983).
9. Poor, W. A. *Differential geometric structures*. McGraw-Hill, (1981).
10. Sasaki, S. On the differential geometry of tangent bundles of Riemannian manifolds. *Tohoku Math. J.*, **10**, (1985).
11. Yano, K. and Ishihara, S. *Tangent and cotangent bundles. Differential geometry*. Marcel Dekker Inc., New York, (1973).