ESTIMATING THE MEAN OF INVERSE GAUSSIAN DISTRIBUTION WITH KNOWN COEFFICIENT OF VARIATION UNDER ENTROPY LOSS

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Abstract

An estimation problem of the mean μ of an inverse Gaussian distribution $IG(\mu, c^{-2}\mu)$ with known coefficient of variation c is treated as a decision problem with entropy loss function. A class of Bayes estimators is constructed, and shown to include MRSE estimator as its closure. Two important members of this class can easily be computed using continued fractions.

1. Introduction

We assume that the distribution of a random variable X is an inverse Gaussian with known coefficient of variation c, denoted by $IG(\mu, c^{-2}\mu)$, with the probability density function

$$f(x;\mu)=(\frac{\mu}{2\pi c^2})^{\frac{1}{2}}x^{\frac{3}{2}}\exp\{\frac{(x-\mu)^2}{2c^2\mu x}\};\quad x>0,\mu>0,c>0. \tag{1.1}$$

The distribution is known as the first passage time distribution of a Brownian motion process. Because of the inverse relationship between the cumulant generating function of the first passage time distribution and that of the Gaussian distribution, Tweedie [10] proposed the name inverse Gaussian for the distribution. When the population mean is equal to unity, the distribution is often referred to as the standard wald distribution. A good summary of the basic properties of the distribution can be found in Folks and Chhikara [3]. In this case $E(X)=\mu$ and $V(X)=c^2\mu^2$, with the coefficient of variation c.

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Let $X_1, X_2, \dots, X_n, (n>3)$ be a random sample of size n from this distribution. Then $(\bar{X}, V)$ form a pair of independent, sufficient but not complete statistics for μ, where $\bar{X}=\frac{1}{n}\sum_{i=1}^{n}X_i$ and $V=\sum_{i=1}^{n}(\frac{1}{X_i}-\frac{1}{\bar{X}})$. Now $\bar{X}$ is distributed as $IG(\mu, nc^{-2}\mu)$ and $\mu c^{-2}V$ is a Chi-Square variate with $(n-1)$ degrees of freedom. The statistic $b=\bar{X}V$ is ancillary with respect to μ.

The family of distributions (1.1) is invariant under the scale changes, and μ is actually a scale parameter, this fact suggests looking at estimators which are equivariant under changes of scale $x_i \rightarrow ax_i$, $a>0$. The estimator $\delta(x_1, x_2, \dots, x_n)$ is scale equivariant if $\delta(ax_1, ax_2, \dots, ax_n)=a\delta(x_1, x_2, \dots, x_n)$, $\forall a>0$, and all $x_1, \dots, x_n$.

If one considers the loss functions, which are invariant under the scale changes, then one can construct scale equivariant estimators.

In this paper, we consider the Kullback-Leibler information number (or the entropy distance) between two distributions of an n-independent inverse Gaussian variable. Kullback [8] described this quantity as the mean information from the likelihood function $f(X, \theta)$ against

$f(X, d)$. This loss is also equivalent to the minimum discrimination information statistic in Kullback [8]. The resulting loss is defined as

$$L(\mu, d) = E_\mu \{\ln [\Pi_{i=1}^n f_\mu(X_i) / \Pi_{i=1}^n f_d(X_i)]\}$$

$$= \frac{n}{2} \ln\left(\frac{\mu}{d}\right) - \frac{1}{2c^2} E_\mu \left[\frac{\sum_{i=1}^n X_i}{\mu} + \mu \sum_{i=1}^n \frac{1}{X_i} - \frac{\sum_{i=1}^n X_i}{d} - d \sum_{i=1}^n \frac{1}{X_i}\right]$$

$$= \frac{n}{2}\left[\frac{d}{\mu} - \ln\frac{d}{\mu} - 1\right] + \frac{n}{2c^2}\left[\frac{d}{\mu} + \frac{\mu}{d} - 2\right] \qquad (1.2)$$

and so this measure is a linear combination of two loss functions, and is called an extended unseparable loss function. For the losses $L_1(\mu, d) = \frac{d}{\mu} + \frac{\mu}{d} - 2$ and

$$L_2(\mu, d) = \left(\frac{d}{\mu} - 1\right)^2, \qquad (1.3)$$

the Best Invariant estimator and the Bayes estimator were obtained by Hirano and Iwase [6] and Joshi and Shah [7]. In this paper we obtain the Bayes and minimax estimator under the scale invariant loss

$$L(\mu, d) = \frac{d}{\mu} - \ln\frac{d}{\mu} - 1, \qquad (1.4)$$

which is called the entropy loss function.
Note that the MLE of μ is

$$\hat{\mu}_{MLE} = \frac{c^2 \bar{X}}{2T}\{1 + (1 + \frac{4T}{c^4})^{\frac{1}{2}}\}$$

where $T = 1 + \frac{\bar{X} V}{n}$, and in this case $\hat{\mu}_{MLE}$, $a_1 \bar{X}_1$ and $a_2 V^{-1}$ are all scale equivariant estimators.

2. Minimum Risk Scale Equivariant Estimator

We start by noting that to find an MRE estimator, we need consider only non-randomized rules $\delta(\bar{X}, V)$ based on the sufficient statistics [9]. Let $\delta(\bar{X}, V)$ be a scale equivariant estimator, then

$$\delta(\bar{X}, V) = \frac{1}{a}\delta(a\bar{X}, a^{-1}V), \quad \forall\ a > 0.$$

Letting $a = \sqrt{\bar{X}^{-1} V}$, then all scale equivariant estimators have to be expressed in the form

$$\delta(\bar{X}, V) = \sqrt{U}\,\phi(B),$$

where $U = \bar{X}V^{-1}$, and $B = \bar{X}V$. There exists at least one estimator of this form which has finite risk under the desired loss function.

The joint distribution of B and $\sqrt{U}\,\phi(B)/\mu$ is independent of μ, so the risk of any equivariant estimator $\sqrt{U}\,\phi(B)$ with finite risk has the representation

$$R(\sqrt{U}\,\phi(B)) = E_\mu\left[\frac{\sqrt{U}\,\phi(B)}{\mu} - \ln\left(\frac{\sqrt{U}\,\phi(B)}{\mu}\right) - 1\right]$$

$$= E\,[E_{\mu=1}(\sqrt{U}\,\phi(b) - \ln\sqrt{U}\,\phi(b) - 1 \mid b)]. \quad (2.1)$$

It follows that the minimum risk scale equivariant estimator $\sqrt{U}\,\phi^*(b)$, if it exists, must satisfy

$$E_{\mu=1}\,[(\sqrt{U}\,\phi^*(b) - \ln\sqrt{U}\,\phi^*(b) - 1) \mid b] =$$
$$\min_{\phi} E_{\mu=1}\,[(\sqrt{U}\,\phi(b) - \ln\sqrt{U}\,\phi(b) - \ln\sqrt{U}\,\phi(b) - 1) \mid b] \quad (2.2)$$

Using (2.2), it is easily shown that

$$\phi^*(b) = [E(\sqrt{U} \mid b)]^{-1}$$

$$= \frac{\int_0^\infty f(u, b)\,du}{\int_0^\infty \sqrt{u}\,f(u, b)\,du} \qquad (2.3)$$

where $f(u, b)$ is the joint distribution of U and B. So the unique MRE estimator which is recognizable as Pitman-type estimator is

$$\hat{\mu}_{MRE} = \sqrt{u}\,[E(\sqrt{U} \mid b)]^{-1}$$

$$= \frac{\sqrt{u}\int_0^\infty f(u, b)\,du}{\int_0^\infty \sqrt{u}\,f(u, b)\,du}.$$

Now, note that

$$f_{\bar{X};\mu=1}(\bar{x}) = \left(\frac{nc^{-2}}{2\pi\bar{x}^3}\right)^{\frac{1}{2}} e^{-\frac{nc^{-2}(\bar{x}-1)^2}{2\bar{x}}}, \qquad \bar{x} > 0$$

and

$$f_V(v) = \frac{1}{\Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n-1}{2}} c^{n-1}} v^{\frac{n-3}{2}} e^{-\frac{v}{2c^2}}, \quad v > 0$$

and after some calculations, we have

$$f_{\bar{X}V,\bar{X}V}(b,u)=\frac{\sqrt{n}\,e^{nc^{-2}}b^{\frac{n-6}{4}}}{\sqrt{\pi}\,2^{\frac{n+1}{2}}c^{n}\Gamma(\frac{n-1}{2})}u^{-\frac{n+4}{4}}e^{-\frac{n\sqrt{b}}{2c^2}\sqrt{u}-\frac{1}{2c^2}(\sqrt{b}+\frac{n}{\sqrt{b}})\frac{1}{\sqrt{u}}},$$

$u>0, b>0.$

Now using

$$\int_0^\infty x^{f-1}e^{-\frac{\delta}{x}-\gamma x}\,dx=2\left(\frac{\delta}{\gamma}\right)^{\frac{f}{2}}K_f(2\sqrt{\delta\gamma});$$
$$Re(\delta)>0,\ Re(\gamma)>0, \tag{2.4}$$

we have

$$\frac{\int_0^\infty f(u,b)\,du}{\int_0^\infty \sqrt{u}f(u,b)\,du}=\frac{\int_0^\infty u^{\frac{n}{2}-1}e^{-\frac{n\sqrt{b}}{2c^2}\frac{1}{u}-\frac{1}{2c^2}(\sqrt{b}+\frac{n}{\sqrt{b}})u}\,du}{\int_0^\infty u^{\frac{n}{2}-2}e^{-\frac{n\sqrt{b}}{2c^2}\frac{1}{u}-\frac{1}{2c^2}(\sqrt{b}+\frac{n}{\sqrt{b}})u}\,du}$$

$$=\frac{2\left(\frac{n\sqrt{b}/2c^2}{(\sqrt{b}+\frac{n}{\sqrt{b}})/2c^2}\right)^{\frac{n}{4}}K_{\frac{n}{2}}\left(2\sqrt{\frac{n\sqrt{b}}{2c^2}\left(\frac{\sqrt{b}+\frac{n}{\sqrt{b}}}{2c^2}\right)}\right)}{2\left(\frac{n\sqrt{b}/2c^2}{(\sqrt{b}+\frac{n}{\sqrt{b}})/2c^2}\right)^{\frac{n}{4}-\frac{1}{2}}K_{\frac{n}{2}-1}\left(2\sqrt{\frac{n\sqrt{b}}{2c^2}\left(\frac{\sqrt{b}+\frac{n}{\sqrt{b}}}{2c^2}\right)}\right)}$$

$$=\left(\frac{nb}{b+n}\right)\frac{K_{\frac{n}{2}}\left(\frac{\sqrt{n(b+n)}}{c^2}\right)}{K_{\frac{n}{2}-1}\left(\frac{\sqrt{n(b+n)}}{c^2}\right)},\quad b=\bar{X}V$$

where, $K_f(.)$ is modified Bessel function of order f [5], so

$$\mu_{MRE}=\sqrt{\bar{X}V}^{-1}\left(\frac{n\bar{X}V}{n+\bar{X}V}\right)^{\frac{1}{2}}\frac{K_{\frac{n}{2}}\left(\frac{\sqrt{n(\bar{X}V+n)}}{c^2}\right)}{K_{\frac{n}{2}-1}\left(\frac{\sqrt{n(\bar{X}V+n)}}{c^2}\right)}$$

$$=\frac{\bar{X}}{\sqrt{T}}\frac{K_{\frac{n}{2}}(\frac{n}{c^2}\sqrt{T})}{K_{\frac{n}{2}-1}(\frac{n}{c^2}\sqrt{T})}. \tag{2.5}$$

It must be noted that from the recurrence relation $K_{v-1}(z)-K_{v+1}(z)=-\frac{2v}{z}K_v(z)$, the estimator $\hat{\mu}_{MRE}$ is rewritten for computational purposes by using a continued fraction as follows. For $n=2k+1$, $k=1,2,\ldots$

$$\hat{\mu}_{MRE}=\frac{\bar{X}}{\sqrt{T}}\left\{\frac{c^2(1-\frac{2}{n})}{\sqrt{T}}+\cfrac{1}{\cfrac{c^2(1-\frac{4}{n})}{\sqrt{T}}+\cfrac{1}{\cfrac{c^2(1-\frac{6}{n})}{\sqrt{T}}+\ldots+\cfrac{1}{\cfrac{c^2(1-\frac{n-1}{n})}{\sqrt{T}}+1}}}\right\}$$

For $n=2k+2$, $k=1,2,\ldots$

$$\hat{\mu}_{MRE}=\frac{\bar{X}}{\sqrt{T}}\left\{\frac{c^2(1-\frac{2}{n})}{\sqrt{T}}+\cfrac{1}{\cfrac{c^2(1-\frac{4}{n})}{\sqrt{T}}+\cfrac{1}{\cfrac{c^2(1-\frac{6}{n})}{\sqrt{T}}+\ldots+\cfrac{1}{\cfrac{c^2(1-\frac{n-2}{n})}{\sqrt{T}}+\cfrac{K_0(n\sqrt{T}/c^2)}{K_1(n\sqrt{T}/c^2)}}}}\right\}.$$

Numerical values of $K_0(x)$ and $K_1(x)$ for $x=0.1(0.1)20$ are calculated in Abromowitz and Stegun [1]. It is easily shown that for large n the estimator $\hat{\mu}_{MRE}$ is asymptotically equal to the MLE, $\hat{\mu}_{MLE}$.

3. Bayes Estimator

Since μ is a scale parameter, the following density function was considered by Joshi and Shah [7].

$$\psi_{\alpha,\beta,p}(\mu)=\begin{cases}d\mu^{p}e^{-(\alpha\mu+\frac{\beta}{\mu})} & \text{if } \alpha,\beta\geq 0, p\in R\\ 0 & \text{otherwise}\end{cases} \tag{3.1}$$

This is a broader family of distributions which has conjugate prior ($p=\frac{j}{2}$), inverted Gamma prior ($p<0$ & $\alpha=0$), Gamma prior ($p>0$ & $\beta=0$) and vague prior ($p<0$ & $\alpha=\beta=0$) as the particular cases, so the Bayes estimator under entropy loss is

$$\hat{\mu}_{Bayes}=\frac{1}{E[\mu^{-1}|\bar{X},V]}$$

$$\frac{\int_0^\infty \mu^{\frac{n}{2}+p}e^{-(\mu S_1+\frac{S_2}{\mu})}d\mu}{\int_0^\infty \mu^{\frac{n}{2}+p-1}e^{-(\mu S_1+\frac{S_2}{\mu})}d\mu}$$

where

$$S_1=\frac{n}{2c\bar{X}}[1+\frac{b}{n}+\frac{2c^2\bar{X}\alpha}{n}]$$

and

$$S_2 = \frac{n\overline{X}}{2c^2}\left[1+\frac{2c^2\beta}{n\overline{X}}\right].$$

Using (2.4)

$$\hat{\mu}_{Bayes} = \frac{2\left(\frac{S_2}{S_1}\right)^{\frac{\frac{n}{2}+p+1}{2}} K_{\frac{n}{2}+p+1}(2\sqrt{S_1S_2})}{2\left(\frac{S_2}{S_1}\right)^{\frac{\frac{n}{2}+p}{2}} K_{\frac{n}{2}+p}(2\sqrt{S_1S_2})}$$

$$= \left(\frac{S_2}{S_1}\right)^{\frac{1}{2}} \frac{K_{\frac{n}{2}+p+1}(2\sqrt{S_1S_2})}{K_{\frac{n}{2}+p}(2\sqrt{S_1S_2})} \qquad (3.2)$$

Now, we can easily see that when $p = -1$, $\alpha \to 0$ and $\beta \to 0$ (i.e. for vague prior $\frac{1}{\mu}$), $\hat{\mu}_{MRE} = \hat{\mu}_{Bayes}$. That is for the entropy loss function the MRE rule $\hat{\mu}_{MRE}$ is the limit, $\lim_{\alpha\to 0\,\beta\to 0} \hat{\mu}_{Bayes}$ of the Bayes rule against the priori $\psi_{0,0,-1}(\mu)$. It is easily seen that the limit as $\alpha \to 0$ and $\beta \to 0$ exists and can be taken inside the integral in (3.2). (And by using the type of argument implied by the main theorem of Farrell [2], it can be shown that $\hat{\mu}_{MRE}$ is admissible Pitman-type estimator).

4. General Minimum Risk Scale Equivariant Estimator

In this section, we give a general form (for any probability density function with scale parameter) of the MRE estimator when the loss function is the entropy loss function. Let $X_1, \ldots, X_n$ have a joint probability density function

$$\frac{1}{\theta^n} f\left(\frac{x_1}{\theta}, \frac{x_2}{\theta}, \ldots, \frac{x_n}{\theta}\right); \quad \theta>0 \qquad (4.1)$$

where θ is an unknown scale parameter. Then

Proposition 4.1. The minimum risk scale equivariant estimator of θ under the entropy loss function (the so called Pitman-type estimator of θ) is

$$\frac{\int_0^\infty t^{n-1} f(x_1t, \ldots, x_nt)\, dt}{\int_0^\infty t^n f(x_1t, \ldots, x_nt)\, dt} \qquad (4.2)$$

Proof. Let $X = (X_1, \ldots, X_n)$ be distributed according to (4.1) and $Z = (Z_1, \ldots, Z_n)$ with $Z_i = \frac{X_i}{X_n}$, $i = 1, \ldots, n-1$, and $Z_n = \frac{X_n}{|X_n|}$. Suppose that there exists a scale equivariant estimator δ_0 of θ with finite risk. **Then under the entropy loss function (1.4) an MRE estimator of θ is** $\delta^*(X) = \frac{\delta_0(X)}{\omega^*(z)}$, where $\omega^*(z)$ is given by

$$\omega^*(z) = E_{\theta=1}[\delta_0(X)|Z=z].$$

Let $\delta_0(X) = X_n$. To compute $E_{\theta=1}[X_n|Z=z]$ which exists, make the one to one transformation $Z_i = \frac{X_i}{X_n}$, $i = 1, \ldots, n-1$ and $U = X_n$. Then

$$f_{Z_1,\ldots,Z_{n-1},U}(z_1,\ldots,z_{n-1},u) = u^{n-1} f(uz_1,\ldots,uz_{n-1},u)$$

and

$$f_{U|Z=z}(u) = \frac{u^{n-1} f(uz_1,\ldots,uz_{n-1},u)}{\int_0^\infty v^{n-1} f(vz_1,\ldots,vz_{n-1},v)\, dv}.$$

Hence,

$$E_{\theta=1}[X_n|Z_1=z_1,\ldots,Z_{n-1}=z_{n-1}] = \frac{\int_0^\infty v^n f(vz_1,\ldots,vz_{n-1},v)\, dv}{\int_0^\infty v^{n-1} f(vz_1,\ldots,vz_{n-1},v)\, dv}.$$

Now, let $v = tx_n$, then we have

$$E_{\theta=1}[X_n|Z=z] = \frac{x_n\int_0^\infty t^n f(x_1t,\ldots,x_nt)\, dt}{\int_0^\infty t^{n-1} f(x_1t,\ldots,x_nt)\, dt}.$$

So,

$$\frac{x_n}{E_{\theta=1}[X_n|Z=z]} = \frac{\int_0^\infty t^{n-1} f(x_1t,\ldots,x_nt)\, dt}{\int_0^\infty t^n f(x_1t,\ldots,x_nt)\, dt}.$$

This completes the proof.

Gleser and Healy [4] gave scale equivariant estimator, $\hat{\theta}_{MRE}$, of the mean θ for the $N(\theta, a\theta^2)$ with minimum risk for all θ when the loss function is the squared error. Hirano and Iwase [6] derived this estimator for the scale equivariant loss $\frac{d}{\theta} - \frac{\theta}{d} - 2$. From the above proposition, the minimum risk scale equivariant estimator of θ, which is positive and a scale parameter, is given by

$$\hat{\theta}_{MRE} = (a^{-1}nS_2')^{-1/2}\frac{I_{n-1}(B)}{I_n(B)} \tag{4.3}$$

where

$$B = a^{-1}n\bar{X}(a^{-1}nS_2')^{-\frac{1}{2}},\quad S_2' = \frac{1}{n}\sum_{i=1}^{n}X_i^2$$

and

$$I_m(a) = \int_0^\infty x^m e^{-\frac{(x-a)^2}{2}}dx$$

when the loss is the entrophy loss function (1.3). In the normal case Gleser and Healy [4] showed that

$$\frac{n}{2a(n+1)}\{-\bar{X} + (\bar{X}^2 + 4a(n+1)n^{-1}S_2')^{\frac{1}{2}}\} \le \hat{\theta}_l \le \hat{\theta}_{MLE}$$

also Hirano and Iwase [6] showed that

$$\hat{\theta}_{MLE} \le \tilde{\theta} \le \frac{n}{2a(n-2)}\{-\bar{X} + (\bar{X}^2 + 4a(n-2)n^{-1}S_2')^{\frac{1}{2}}\}$$

where $\tilde{\theta}$ was the MRE estimator under the loss (1.3). In our case it can be easily shown that

$$\hat{\theta}_{MLE} \le \hat{\theta}_{MRE} \le \frac{n}{2a(n-1)}\{-\bar{X} + (\bar{X}^2 + 4a(n-1)n^{-1}S_2')^{\frac{1}{2}}\} \tag{4.4}$$

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