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ON THE INFINITE ORDER MARKOV PROCESSES

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Abstract

The notion of infinite order Markov process is introduced and the Markov property of the flow of information is established.

Introduction

Let $\{X_t, t \geq 0\}$ be a stochastic process and define:

$$\overset{-}{\Sigma}_t(X) = \bigcap_n \sigma\{X_s, s < t + \frac{1}{n}\}$$

$$\overset{+}{\Sigma}_t(X) = \bigcap_n \sigma\{X_s : s > t - \frac{1}{n}\}$$

$$\Gamma_t(X) = \bigcap_n \sigma\{X_s : |s - t| < \frac{1}{n}\}.$$

Germ field Markov property [3] requires that for each $t > 0$ given Γ_t, the two σ- fields $\Sigma_t^-(X)$ and $\Sigma_t^+(X)$ be independent. Σ_t^-, Σ_t^+, and Γ_t are called the past, future, and the present of the process, respectively. The collection $\{\Gamma_t\}$ is called the flow of information.

In this paper, the concept of *N - ple* Markov property has been generalized. Here the flow of information is generated by a family of stochastic processes $\{Y_u(t), t \geq 0\}$, $u \in R$. In order to make sure that the flow of information was as small as possible, the generators of the flow of information were required to be linearly independent. Finally, we prove that the flow has Markov property. As in [1] and [2], a martingale representation of the process would be of interest.

Keywords: Flow of information; Gaussian processes; Markov Property

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Definitions and Results

Let $X = \{X_t : t \geq 0\}$ be a Gaussian mean 0 and continuous in quadratic mean process on some probability space (Ω, F, P). For each $t \geq 0$, let $\{Y_u(t), u \in R\}$ be another Gaussian process on (Ω, F, P) with mean 0 and continuous on both parameters u and t.

Definition

We say that the collection of processes $\{Y_u(t), u \in R\}$. $t \geq 0$ is **free** if for each t, n, and for each real $u_1, \ldots, u_n$, the random variables $Y_{u1}(t), \ldots, Y_{un}(t)$ are linearly independent as elements of $L^2(\Omega, G_t, P)$, where

$$G_t = \bigcap_n \sigma\{X_s : |t - s| < \frac{1}{n}\}$$

We have the following definition of infinite order Markov process.

Definition

Let $X = \{X_t, t \geq 0\}$ be a Gaussian process with mean 0 and continuous in the mean. We say X has infinite order Markov property with respect to

$$\{Y_u(t), u \in R, t \geq 0\}.$$

if,

(i) for each $t \geq 0$, $\{Y_u(t), u \in R\}$ is a free mean zero Gaussian process in $L^2(\Omega, G_t, P)$, $\{Y_u(t), u \in R\}$ is assumed to be continuous in the mean on both parameters u and t.

(ii) $\Sigma_t^- \perp \Sigma_t^+ | \Gamma_t$, i.e. given Γ_t; Σ_t^- and Σ_t^+ are conditionally independent [3], here

$$\bar{\Sigma}_t = \bigcap_n \sigma\{X_s : s < t + \frac{1}{n}\}$$

$$\overset{+}{\Sigma}_t = \bigcap_n \sigma\{X_s : s > t - \frac{1}{n}\}$$

$$\Gamma_t = \sigma\{Y_u(t),\ u \in R\}$$

(iii) $\{X(t), t \geq 0\}$ and $\{Y_u(t), u \in R . t \geq 0\}$ are jointly Gaussian.

Theorem 1

Let $\{X_t, t \geq 0\}$ be an infinite order Markov process with respect to $\{Y_u(t), u \in R\}, t \geq 0$ then we have the following representation:

$$X_t = \int Y_u(t)\, G_t(du),$$

where for each $t \geq 0$, G_t is a finite Borel measure for which $\int Y_u(t)\, G_t(du)$ is well defined.

Proof. From Markov property, for each $t > s \geq 0$, we get

$$E(X_t \mid \bar{\Sigma}_s) = E(X_t \mid \Gamma_s).$$

Since in the case of Gaussian processes the conditional expectations are orthogonal projections, for each $t > s$ we have

$$E(X_t \mid \bar{\Sigma}_s) = P^{X_t}_{H_s^-}$$

where $H_s^- = \bigcap_{n \geq 1} sp\{X_u : u < s + \frac{1}{n}\}$ and P^X_H stands for the projection of X into the subspace H. From Markov property we get

$$E(X_t \mid \bar{\Sigma}_s) = P^{X_t}_{H_s},$$

where $H_s = sp\{Y_u(s), u \in R\}$. Now we observe that

$sp\{Y_u(s), u \in R\} = \{\int Y_u(s)\, G(du)$: G a finite Borel measure$\}$.

To prove this equality we observe that the right-hand side is a subset of the left-hand side, moreover by taking

$$G_u(dv) = \begin{cases} 0 & if\, v \neq u \\ 1 & if\, v = u, \end{cases}$$

we see that $Y_u(s) = \int Y_v(s)\, G_u(dv)$, which proves that the left-hand side is a subset of the right-hand side. Now

$$\begin{aligned} X_t &= E(X_t \mid H_t^-) \\ &= E(X_t \mid \Gamma_t) \\ &= P^{X_t}_{H_t} \\ &= \int Y_u(t)\, G_t(du)\, du, \end{aligned}$$

for some finite Borel measure G_t. This completes the proof.

Theorem 2

Let $\{X_t, t \geq 0\}$ be an infinite order Markov process with respect to $\{Y_u(t), u \in R, t \geq 0\}$, then

$$\sigma\{Y_u(s) : u \in R, s \leq t\} \perp \sigma\{Y_u(s) : u \in R, s \geq t\} \mid \Gamma_t$$

($A \perp B \mid C$ means that given C, A and B are independent).

Proof. For each n≥1, we have

$$\sigma\{Y_u(s) : u \in R, s < t - \frac{1}{n}\} \subset \sigma\{X(s), s < t\}$$

and

$$\sigma\{Y_u(s) : u \in R, s > t + \frac{1}{n}\} \subset \sigma\{X(s), s > t\}.$$

By assumption we have

$$\sigma\{X(s) : s < t\} \perp \sigma\{X(s) : s > t\} \mid \Gamma_t.$$

Therefore, for each n, we have

$$\sigma\{Y_u(s) : u \in R, s < t - \frac{1}{n}\} \perp \sigma\{Y_u(s) : u \in R, s > t + \frac{1}{n}\} \mid \Gamma_t.$$

Thus,

$$\bigvee_n \sigma\{Y_u(s) ; u \in R, s < t - \frac{1}{n}\} \perp \bigvee_n \sigma\{Y_u(s) : u \in R, s > t + \frac{1}{n}\} \mid \Gamma_t,$$

Since $Y_u(s)$ is continuous in both parameters u and s, we get

$$\sigma\{Y_u(s) : u \in R, s \leq t\} \perp \sigma\{Y_u(s) : u \in R, s \geq t\} \mid \Gamma_t,$$

This completes the proof.

Remark

From Theorems 1 and 2 we get

$$\begin{aligned} E(Y_u(t) \mid Y_v(z)\ v \in R, z < s) &= E(Y_u(t) \mid Y_v(s), v \in R). \\ &= \int Y_v(s)\, g_u(t, s, dv), \end{aligned}$$

for some finite Borel measure g.

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