ON OPTIMAL NOZZLE SHAPES OF GAS-DYNAMIC LASERS

A. R. Bahrampour and M. Radjabalipour*

University of Kerman and the International Center for Science, High Technology and Environmental Sciences, Kerman, Islamic Republic of Iran

Abstract

Pontryagin's principle is used to study the shape of the supersonic part of the nozzle of a carbon dioxide gas-dynamic laser whose gain is maximal. The exact shape is obtained for the uncoupled approximation of Anderson's bimodal model. In this case, if sharp corners are allowed, the ceiling of the supersonic part consists of a slant rectangular sheet followed by a horizontal one; otherwise, a parabolic cylinder joins the two sheets smoothly. Pontryagin's principle reduces the optimal control problem to a multifactor optimal problem of the types treated by S.A. Losev, V.N. Makarov, N.M. Reddy, and V. Shanmugasundaram.

I. Introduction

We are concerned with carbon dioxide gas-dynamic sers whose supersonic parts are axisymmetric about an is Ot such that the point O is at the throat, and the cross-ction of the nozzle with the plane perpendicular to Ot at ch point t is a rectangle of constant width and varying ight. To describe the shape of the nozzle, we choose the thogonal coordinate system O-tyz in which Oy and Oz e perpendicular to the ceiling and the walls of the nozzle, spectively. Note that our analysis is based on the steady ate equation and thus the time is eliminated everywhere. ence, the use of t for a spatial coordinate should not cause ny ambiguity and instead enables us to use the notations f [4, pp. 27-28] in Pontryagin's principle. Thus, through-ut the paper $\dot{z}$ denotes the spatial derivative dz/dt.

Throughout the paper, the gas mixture will be CO_2, N_2, nd H_2O. It is assumed that the nozzle has a quasi 1-imensional steady-state flow with no change in its chemi-al composition. Using an m-vibrational temperature model, each point M of the nozzle we correspond a vector

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m+1} \\ x_{m+2} \end{pmatrix} = (x_1, x_2, \dots, x_{m+1}, x_{m+2})' :$$

$$= (T_1, T_2, \dots, T, A/A^*)' \in \mathbb{R}^{m+2} \qquad (1)$$

whose first m components T_1, T_m are the various vibrational temperatures of the gas, whose $(m+1)st$ component T is the translational temperature of the gas, and whose $(m+2)nd$ component is the area ratio A/A^* at M, where A^* is the throat area and A is the area of the cross-section passing through M. Note that the ceiling and the floor of the supersonic part of the nozzle have equations $y = \pm(h/2)x_{m+2}(t)$, $0 \le t \le t_1$, and its walls have the equations $z = \pm w/2$ where h and w are the height and the width of the (rectangular) throat and t_1 is the length of the supersonic part. It follows from our assumption that if M varies on a fixed cross-section perpendicular to Ot at a point t, then the vector (1) remains constant.

Losev-Makarov [8,9,10] studied the optimization of the gain of such lasers by varying the initial temperatures.

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pressure, and composition of the gas mixture, as well as the nozzle geometry. As the nozzle geometry is completely determined by $x_{m+2}(t)$ $(0 \leq t \leq t_1)$, they examined various parametric functions for x_{m+2} to obtain a better gain. Among their tests are wedges of the form $x_{m+2}(t)= 1+\beta t$ and piecewise parabolic nozzles of the form $x_{m+2}(t)= a_it^2 +b_it+c_i$ for $d_{i-1}\leq t\leq d_i$ $(i= 1,2,\ldots,n)$ such that $d_0=0$ and $d_n= t_1$. The examples are all increasing differentiable functions which are concave downward. For other examples, some of which concave upward, see [3,5,11-17] and the references cited therein.

In the present paper, we apply Pontryagin's principle to find an optimal increasing $x_{m+2}(t)$ among piecewise C^1 functions if the oblique shock waves in the active medium are ignored, or among piecewise C^2 functions with downward concavity, otherwise. In the first case, $u= \dot{x}_{m+2}$ is taken as the control; and in the second case, a new component $x_{m+3}= \dot{x}_{m+2}$ is added to the vector (1) and $u= \dot{x}_{m+3}$ is taken as the control. In both cases we assign reasonable initial values to $x_1(0),\ldots,x_{m+1}(0)$ and hence we have

$$x_i(0)= T_i(0), x_{m+2}(0)=1, i=1, 2,\ldots,m+1. \quad (2)$$

To avoid the flow detachment from the nozzle walls, we further assume

$$0 \leq \dot{x}_{m+2} \leq \beta \quad (3)$$

for some positive known constant β [8, pp. 782-783]. (Note that $\dot{x}_{m+2}\geq 0$ follows from the assumption that $x_{m+2}(t)$ is increasing). Condition (3) for the first case determines a set $U= [0, \beta]$ containing the values $u(t)$ of the control u. The same condition imposes restrictions $0\leq x_{m+3}\leq\beta$ in the second case.

Avoiding oblique shock waves in the active medium, the curvature of the nozzle must be restricted. Thus, for the second case we assume

$$-\alpha \leq u= \dot{x}_{m+3}= \ddot{x}_{m+2} \leq 0 \quad (4)$$

for a certain $\alpha >0$ [6, pp. 427-430].

Summing up, we recognize the following different problems.

Problem A. (Ignoring oblique shock waves). Finding an optimal trajectory $x(t) \in \mathbb{R}^{m+2}$ and an optimal control $u(t)=\dot{x}_{m+2}(t) \in [-O, \beta]$ satisfying the initial conditions (2) and a certain equation of motion.

Problem B. (Avoiding oblique shock waves). Finding an optimal trajectory $x(t) \in \mathbb{R}^{m+3}$ and an optimal control $u(t)= \dot{x}_{m+3}(t) \in [-\alpha, 0]$ satisfying the initial conditions (2), the constraints $0 \leq x_{m+3}\leq \beta$, and a certain equation of motion.

Problems with constraints appear because of the complication arising from the junction points at which the trajectories meet or leave the boundary. Fortunately, Problem B can be split into the following simpler Problems B1-B4.

Problem B1. Finding an optimal trajectory $x(t) \in \mathbb{R}^{m+3}$ and an optimal control $u(t)= \dot{x}_{m+3}(t) \in [-\alpha, 0]$ satisfying the initial conditions (2) and a certain equation of motion.

If we are lucky enough to find an optimal solution $x(t) \in \mathbb{R}^{m+3}$ of Problem B1 satisfying $0\leq x_{m+3}(1)\leq x_{m+3}(0)\leq\beta$, then we have found a solution of Problem B. This is because $u\leq 0$ and hence $x_{m+3}(t)$ is decreasing. Otherwise, we proceed to the next two problems.

Problem B2. As in Problem B1 with the extra initial condition

$$x_{m+3}(0)= \beta. \quad (5)$$

Problem B3. As in Problem B1 with the extra end condition

$$x_{m+3}(t_1)= 0. \quad (6)$$

The optimal solution of Problem B is now among the optimal solutions x of Problems B2 and B3 if $0\leq x_{m+3}(t_1)\leq x_{m+3}(0)\leq\beta$. If no such solution exists, we proceed to the next problem.

Problem B4. As in Problem B1 with the extra end conditions (5) and (6).

It is easy to see that if Problem B has a solution it must be among those solutions x of Problems B1-B4 that satisfy $0 \leq x_{m+3}(t_1) \leq x_{m+3}(0) \leq \beta$.

Problem A will be dealt with in section II where a complete solution is obtained for the uncoupled approximation of the Anderson's bimodal model. Problem B is dealt with in section III in a similar manner.

II. Oblique Shock Waves Ignored

In this section, we assume the effects of the oblique shock waves can be ignored in the active medium and hence the nozzle shape may have sharp corners. Thus $x_{m+2}(t)$ is assumed to be piecewise C^1; that is $\dot{x}_{m+2}(t)$ is piecewise continuous and, without loss of generality, left continuous. Later we will see that the equation of motion needed in Pontryagin's principle is described by a system of the form

$$\dot{x}_i = \frac{1}{x_{m+2}} F_i(x_1, \ldots, x_{m+1}),\ i = 1, 2,\ldots, m, \quad (7)$$

$$\dot{x}_{m+1} = \frac{1}{x_{m+2}} [uF_{m+1}(x_1,\ldots,x_{m+1})+F_{m+2}(x_1,\ldots,x_{m+1})], \quad (8)$$

$$\dot{x}_{m+2}= u, \quad (9)$$

where $F_i(i= 1,\ldots, m+2)$ are smooth functions except for F_{m+1} and F_{m+2} which are unbounded at the throat (the point at which the frozen Mach number M_f is equal to 1) [5, p. 1087].

The equations (7)-(9) will be shortened as

$$\dot{x} = f(x,u),\ x(t) \in \mathbb{R}^{m+2},\ u(t) \in [0, \beta]. \tag{10}$$

We are adjusting our notation to that of [4, pp. 23-28] for the application of Pontryagin's principle. Thus, we define

$$e = (t_0, t_1, x(t_0), x(t_1)) \in \mathbb{R}^{2m+6}, \tag{11}$$

where $[t_0, t_1]$ is the interval on which the optimal trajectory $x(t)$ is defined. The function to be minimized is $-g_0(x_1,\ldots, x_{m+1})$, where g_0 is the small signal gain of the active medium.

Hence, as in [4, p. 24], we define the performance index $\phi_1: \mathbb{R}^{2m+6} \to \mathbb{R}$ such that

$$\phi_1(e) := -g_0(x_1(t_1),\ldots, x_{m+1}(t_1)). \tag{12}$$

The fact that the throat is always at t=0 is formalized by

$$\phi_2(e) := t_0 = 0. \tag{13}$$

The initial conditions (2) are formalized by

$$\phi_i(e) := x_{i-2}(t_0) - T_{i-2} = 0 (i = 3,4,\ldots,m+3), \tag{14}$$

$$\phi_{m+4}(e) := x_{m+2}(t_0) - 1 = 0. \tag{15}$$

Thus, we obtain a function $\phi: \mathbb{R}^{2m+6} \to \mathbb{R}^{m+4}$ whose values at e are defined by (12)-(15).

Following the notation of [4], Problem A can be now restated as follows.

II.1. Problem. Let $\mathscr{F}$ be the set of all $(m+3)$-tuples $(x,u) = (x_1 \ldots x_{m+2}, u)$ such that $x_1,\ldots,x_{m+2}$ are piecewise C^1 functions and $u: [t_0, t_1] \to [0, \beta]$ is a piecewise continuous, left continuous function satisfying the equation of motion (10) and the initial conditions (13)-(15). Define $K: \mathscr{F} \to \mathbb{R}$ by $K(x, u) = \phi_1(e)$, where e is as in (11). Find $(x, u) \in \mathscr{F}$ such that K is minimum.

Problem II.1 is a Mayer type problem [4, p. 25]. The solution (x, u) is called an optimal point. The parts x and u are called an optimal trajectory and an optimal control, respectively. It follows from (7)-(9) that x_i $(i = 1,\ldots,m)$ is C^1, and x_{m+1} and x_{m+2} are piecewise C^1 except at $t = 0$, where $\dot{x}_{m+1}$ is unbounded.

A partial solution to Problem II.1 is given by the following theorem.

II.2. Theorem. Let (x, u) be an optimal solution of Problem II.1. Then there exist $\eta \in \{0, 1\}$ and a function $P: [0, t_1] \to \mathbb{R}^{m+2}$ for some $t_1 > 0$ such that

$$\dot{P}_i(t) = -\sum_{j=1}^{m+2} P_j(t)\, \partial f_j(x(t), u(t))/\partial x_i \ (i = 1,\ldots,m+2), \tag{16}$$

$$P_i(t_1) = \eta\, \partial g_0(x_1(t_1),\ldots,x_{m+1}(t_1))/\partial x_i \ (i = 1,\ldots,m+1), \tag{17}$$

$$H(t, x(t), u(t)) = 0\ (0 \le t \le t_1), \tag{18}$$

$$P_{m+2} \equiv 0, \tag{19}$$

$$u(t) = \dot{x}_{m+2}(t) = \beta \text{ if } P_{m+1}(t) f_{m+1}(t) > 0, \tag{20}$$

$$u(t) = \dot{x}_{m+2}(t) = 0 \text{ if } P_{m+1}(t) f_{m+1}(t) < 0, \tag{21}$$

where $H(t,x,u) = \sum_j P_j(t) f_j(x,u)$ is the Hamiltonian of the system. Moreover, if the set

$$\{t \in [0, t_1]: P_{m+1}(t) f_{m+1}(t) = 0,\ u(t) \in \{0, \beta\}\} \tag{22}$$

has no interior point, then x_{m+2} consists of a finite number of line segments with alternating slopes β and 0.

Proof. Let e be as in (11). It follows from [4. pp. 27-28] that there exist a nonzero vector $\lambda \in \mathbb{R}^{m+4}$ with $\lambda_1 \in \{0, -1\}$ and a vector function P: $[0, t_1] \to \mathbb{R}^{m+2}$ satisfying (16) and the following extra conditions:

$$P(t_1)' = \lambda' \phi_{x(1)} =$$

$$\lambda' \begin{bmatrix} -\frac{\partial g_0}{\partial x_1}(x_1(t_1),\ldots,x_{m+1}(t_1)) \ldots -\frac{\partial g_0}{\partial x_{m+1}}(x_1(t_1),\ldots,x_{m+1}(t_1))\ 0 \\ 0 \end{bmatrix} \tag{23}$$

$$H(t_1, x(t_1), u(t_1)) = -\lambda' \phi_{t_1} = 0, \tag{24}$$

$$H(t, x(t), u(t)) = \lambda' \phi_{t_0} + \int_{t_0}^{t} P(s) f_t(s, x(s), u(s))\, ds = \lambda_2, \tag{25}$$

$$\max\{H(t, x(t), u): 0 \le u \le \beta\} = H(t, x(t), u(t)), \tag{26}$$

for all $t \in [0, t_1]$.

It follows from (23)-(25) that (17) and (18) hold for $\eta = -\lambda_1$ and that $P_{m+2}(t_1) = 0$. Using (7)-(9) we conclude from (16) and (18) that

$$\dot{P}_{m+2}(t) x_{m+2}(t) + P_{m+2}(t) \dot{x}_{m+2}(t) = 0$$

Hence $P_{m+2}x_{m+2}$ is constant. Since $x_{m+2}(t)\geq 1$ and $P_{m+2}(t_1)=0$, $P_{m+2}\equiv 0$. This proves (19).

Now (20)-(21) follow from (26), and the last conclusion follows from the fact that $\dot{x}_{m+2}(t)=u(t)$ is piecewise continuous.■

To justify (7)-(8) we follow [18, pp. 198-204] to obtain the relations between $x(t)$ and $u(t)$. Letting $E_i(s)$ be the vibrational energy of mode i at temperature s and $G_i = dE_i/ds$, it follows from [18, p. 204, formula (2.11)] that

$$\dot{x}_i = \frac{E_i(x_{m+1}) - E_i(x_i)}{\tau_i\, v G_i(x_i)} \quad (i=1,2,\ldots,m), \tag{27}$$

where v is the gas velocity. (Recall that $\dot{x}_i = dx_i/dt$ with t denoting the spatial coordinate of the gas molecule, while the variable t in (2.11 of [18] stands for time).

For an expression for E_i see [1, pp. 17-20]. The gas velocity is obtained by

$$v = v(x_1,\ldots,x_{m+1}) = \sqrt{2[H-\gamma(\gamma-1)^{-1}Rx_{m+1} - \sum_{i=1}^{m} E_i(x_i)}, \tag{28}$$

for some constants H, γ and R which is a variation of the energy conservation law $v^2/2+\gamma(\gamma-1)RT+\Sigma_{i=1}^{m}E_i(T_i)=H$ [13, p. 2568]. Finally, τ_i is given by

$$\tau_i = \tau_i(x_1,\ldots,x_{m+2}) = \frac{\delta_i u x_{m+2}}{x_{m+1}} \exp(\gamma_i\, x_{m+1}^{-1/3}), \tag{29}$$

for some constants δ_i and γ_i ($i=1,2,\ldots,m$) [18, p. 204, formula (2.13a)] and [13, p. 2568, formulas (2.24), (2.27)]. Thus (7) holds.

To justify (8), let $M_f = v/\sqrt{\gamma RT}$, and let $a_f = \sqrt{\gamma RT}$. It follows from [5, p. 1087] and (27)-(29) that (8) holds for

$$F_{m+1}(x_1,\ldots,x_{m+1}) = \frac{(\gamma-1)\,x_{m+1}}{1-M_f^2} M_f^2, \tag{30}$$

$$F_{m+2}(x_1,\ldots,x_{m+1}) = \frac{(\gamma-1)x_{m+1}(1-\gamma M_f^2)}{(1-M_f^2)a_f^2} \sum_{i=1}^{m} G_i(x_i)F_i(x_1,\ldots,x_{m+1}) \tag{31}$$

where $F_1,\ldots,F_m$ are as in (7). Note that $F_1,\ldots,F_{m+2}$ are smooth functions except for F_{m+1} and F_{m+2} being possibly unbounded at the throat, the only place at which the frozen Mach number M_f is equal to 1.

Now we sharpen the conclusion of Theorem II.2 in case that m=2 and that an uncoupled approximation of the Anderson's model is used. We assume the gas has a frozen nozzle flow [1, pp. 48-52]. In this case we assume, as in [1], that $\dot{T}_1$ and $\dot{T}_2$ are negligible with respect to $\dot{T}$. Also, $G_1(T_1)$ and $G_2(T_2)$ are very small compared to $\gamma(\gamma-1)^{-1}R$. Thu by differentiating the energy conservation law

$v^2/2 + \gamma(\gamma-1)^{-1}RT + E_1(T_1) + E_2(T_2) = H$, we have
$v\dot{v}+\gamma(\gamma-1)^{-1}R\dot{T} \approx v\dot{v}+\gamma(\gamma-1)^{-1}R\dot{T}+G_1(T_1)\dot{T}_1+G_2(T_2)\dot{T}_2=0.$

Therefore, we can assume v is a function of $x_3 = T$ alone Now, it follows from the mass and momentum conserva tion laws $\rho vA/A^*=Q$ and $\rho dv + dP=0$, together with the gas state equation $P=\rho RT$, that

$$\dot{T} = \left(\frac{A^*}{A}\right)\frac{(\gamma-1)M_f^2 T}{(1-M_f^2)}\frac{d(A/A^*)}{dt}.$$

Using our notation, we have

$$\dot{x}_3 = \frac{1}{x_4}\frac{(\gamma-1)M_f^2 x_3}{1-M_f^2}u.$$

Thus, in view of (30)-(31), $F_4\equiv 0$ and F_3 is a function $\widetilde{F}_3$ of x_3 alone. Hence, it follows from (8), that $dx_3/\widetilde{F}_3(x_3)=dx_4/x_4$ and therefore

$$x_4 = \exp\int_{x_3(0)}^{x_3} [\widetilde{F}_3(s)]^{-1}\,ds. \tag{32}$$

Replacing x_4 in (7)-(8) and noting that v is a function of x_3 alone yields the following new equation of motion:

$$\dot{x}_i = F_i(x_i, x_3) \quad (i=1,2), \tag{33}$$

$$\dot{x}_3 = uF_3(x_3), \tag{34}$$

$$\dot{x}_4 = u. \tag{35}$$

It follows from Theorem II.2 that if (x,u) is an optimal solution of Problem II.1, then there exists a function $P: [0, t_1] \rightarrow \mathbb{R}^4$ such that

$$\dot{P}_i = -P_i\partial F_i/\partial x_i \quad (i=1,2), \tag{36}$$

$$\dot{P}_3 = -P_1\partial F_1/\partial x_3 - P_2\partial F_2/\partial x_3 - P_3 u\frac{dF_3}{dx_3}, \tag{37}$$

$$P_4 \equiv 0 \equiv H(t, x(t), u(t)) = P_1(t)F_1(x_1(t), x_3(t)) + P_2(t)F_2(x_2(t), x_3(t)) + P_3(t)F_3(x_3(t))\,u(t). \tag{38}$$

Letting $L_i = \tau_i v$, it follows from (36) that

$$\frac{\dot P_i}{P_i}=\frac{1}{L_i(x_3)}+\frac{1}{G_i(x_i(t))}\frac{d}{dt}G_i(x_i(t)) \quad (i=1,2). \tag{39}$$

Then, if $P_i(t'_i)\neq 0$ for some $t'_i\in[0,t_1]$, then

$$P_i(t)=P_i(t'_i)\frac{G_i(x_i(t))}{G_i(x_i(t'_i))}\exp\left(\int_{t'_i}^{t}\frac{dt}{L_i(x_3(t))}\right) \tag{40}$$

or $i=1,2$. Since $G_i(x_i)\neq 0$ for all x_i [13, p. 2568], it follows hat $P_i(t)\neq 0$ for all $t\in[0,t_1]$ if and only if $P_i(t)\neq 0$ for some $\in[0,t_1]$ $(i=1,2)$. Thus (40) holds if $t'_i=0$ $(i=1,2)$.

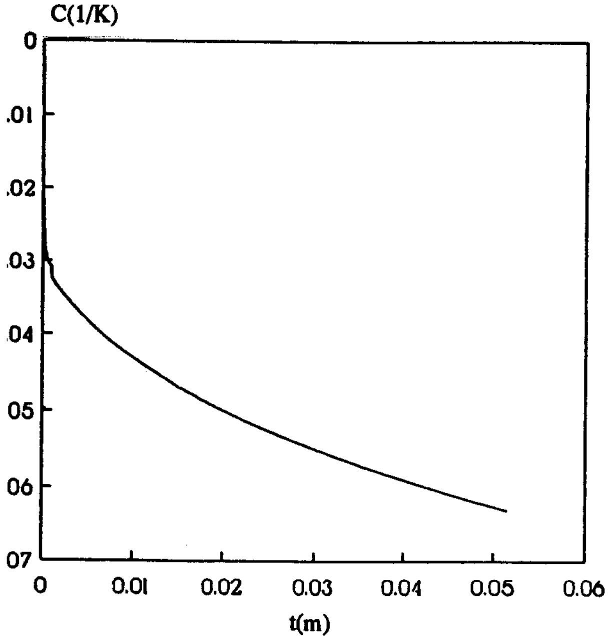
There are certain conditions on F_i, G_i which automatically hold in the supersonic part of an optimal nozzle; therwise, we have no population inversion. Some of them re as follows:

$$F_i<0, \quad (i=1,2,3)$$

$$\frac{G_1(x_3(t))}{E_1(x_3(t))-E_1(x_1(t))}-\frac{G_2(x_3(t))}{E_2(x_3(t))-E_2(x_2(t))}+\frac{\gamma_2-\gamma_1}{3x_3(t)^{4/3}}<0$$

See [1,7,16,17] and Fig. 1). We will see in the proof of heorems II.4 and III.5 that $P_1<0$ and hence $P_1F_1>0$ in the upersonic part. The following lemma is based on the bove observations.

II.3. Lemma. Let P_1, P_2, F_1 and F_2 be as above and define



gure 1. $C(t)=\frac{G(x_1)}{e_1(x_3)-e_1(x_1)}-\frac{G(x_2)}{e_2(x_3)-e_2(x_2)}-\frac{d}{dx_3}\ln\frac{\tau_2}{\tau_1}$ versus (t) for a wedge nozzle with 15° half angle

$$\tilde Q(t)=-P(t)F_1(x_1(t),x_3(t))-P_2(t)F_2(x_2(t),x_3(t)) \tag{41}$$

for $t\in[0,t_1]$. Assume $\tilde Q(t')=0$ for some t', and suppose $\dot x_4(t'-)>0$ (resp. $\dot x_4(t'+)>0$). Then there exists $\varepsilon>0$ such that $\tilde Q(t)>0$ (resp. $\tilde Q(t)<0$) for all $t\in(t-\varepsilon,t')$ (resp. $t\in(t',t'+\varepsilon)$).

Proof. For convenience we may write $j(t)$ for $j(x(t))$ if j is a function of $x=(x_1,\dots,x_4)$. It follows from the hypothesis that

$$P_2(t')=-P_1(t')F_1(t')/F_2(t'). \tag{42}$$

(Note that $F_i(t)<0$ for all $t\in[0,t_1]$, $i=1,2,3$). It follows that

$$\tilde Q(t)=-P_1(t')F_1(t')Q(t), \tag{43}$$

where

$$Q(t)=\frac{q_1(t)}{q_1(t')}-\frac{q_2(t)}{q_2(t')}, \tag{44}$$

$$q_i(t)=\frac{\Delta_i(t)}{L_i(t)}\exp\left(\int_0^t\frac{d\tau}{L_i(\tau)}\right), \tag{45}$$

and $\Delta_i(t)=E_i(x_3(t))-E_i(x_i(t))<0$ for all $t\in[0,t_1]$. (Note that $P_1F_1>0$ and hence $\tilde Q Q<0$ on $[0,t_1]$.)

Now, for all $t\in[0,t_1]$ and for $i=1,2$,

$$\dot Q(t)=\dot x_4(t)F_3(t)\sum_{i=1}^{2}(-1)^{i+1}\frac{q_i(t)}{q_i(t')}\left[\frac{G_i(x_3(t))}{\Delta_i(t)}-\frac{1}{L_i(t)}\frac{dL_i}{dx_3}(t)\right]. \tag{46}$$

Hence, if $\dot x_4(t'-)>0$, then

$$\dot Q(t'-)=\dot x_4(t'-)F_3(t')\left[\frac{G_1(x_3)(t')}{\Delta_1(t')}-\frac{G_2(x_3)(t')}{\Delta_2(t')}+\frac{\gamma_2-\gamma_1}{3x_3(t')^{4/3}}\right], \tag{47}$$

and hence $\dot Q(t'-)>0$. (See the paragraph preceding the lemma). Therefore, there exists $\varepsilon>0$ such that $\tilde Q(t)<0$ for all $t\in(t'-\varepsilon,t')$. A similar argument in case $\dot x_4(t'+)>0$ completes the proof.■

We are now ready to state and prove our result for the uncoupled frozen flow approximation of Anderson's bimodal model.

II.4. Theorem. Assume (x,u) is an optimal solution of Problem II.1 in which the equation of motion is changed to (33)-(35). Then

$$x_4(t)=\begin{cases} 1+\beta t, 0\le t\le t_2, \\ 1+\beta t_2, t_2\le t\le t_1, \end{cases} \quad (48)$$

for some $t_2 \in (t_0, t_1)$.

Proof. Let $\lambda'=(\lambda_1,\ldots,\lambda_6)$ be as in the proof of Theorem II.2. If $\lambda_1=0$, then $P_i(t_1)=0$, $i=1,2,3$. It follows from (40) that $P_i \equiv 0$ $(i=1,2)$. By (37)-(38), $P_3 u\equiv 0$ and $\dot{P}_3 \equiv 0$. Thus $P_3 \equiv 0$. By [4, p. 27, Formulas (5.4), (5.6)], $\lambda=0$; a contradiction. Thus, our problem is not abnormal; i.e. $\lambda_1=1$ [4, p. 28]. By [13, p. 2572], g_0 is strictly decreasing with respect to x_1 and hence $\partial g_0/\partial x_1<0$. Thus $P_1(t_1)<0$ and hence $P_1(t)$ for all $t\in[0, t_1]$. By (20)-(21), $P_3(t)F_3(t)u(t)\ge 0$, and thus $P_2(t)=[-P_1(t)F_1(t)-P_3(t)F_3(t)u(t)]/F_2(t)>0$.

Assume, if possible, that $P_3(t)=0$ for all t in some open interval $(\xi,\zeta)\subset(0,t_1)$ and that $u(t'')>0$ for some $t''\in(\xi,\zeta)$. Then $\widetilde{Q}(t)$ is identically 0 and u is positive and continuous on some open interval (ξ',ζ'), where $\widetilde{Q}$ is as in Lemma II.3; a contradiction. We conclude that if P_3 is identically 0 on some open interval, then so is u on the same interval. Therefore, in the light of (20)-(21) and the fact that $f_3(t)<0$ for all $t\in[0, t_1]$,

$$u(t)=\begin{cases} \beta \ , & P_3(t)<0 \\ 0 \ , & P_3(t)\ge 0. \end{cases} \quad (49)$$

Since u is piecewise continuous and left continuous, there exists a partition $\{c_0=0<c_1<\ldots<c_k=t_1\}$ such that u is constant on each subinterval $(c_{i-1},c_i]$ $(i=1,2,\ldots,k)$ and discontinuous at $c_1,c_2,\ldots,c_{k-1}$. Now, fix $i=1,2,\ldots,k-1$ and assume, if possible, that $u(t)=\beta$. Then there exists $\varepsilon>0$ such that $u(t)=\beta$ and $\widetilde{Q}(t)<0$ for all $t\in(c_i, c_i+\varepsilon)$, where $\widetilde{Q}$ is as in Lemma II.3 with $t'=c_i$. Then it follows from (38) that $P_3(t)>0$ and hence $u(t)=0$ for all $t\in(c_i, c_i+\varepsilon)$, a contradiction. Thus $u(c_i+)=0$. Hence $k\le 2$. If $k=1$; the supersonic part of the nozzel is either a channel or a wedge. It is known that the channel can produce no population inversion. Thus, ignoring the channel, we have $k\le 2$ and

$$u(t)=\begin{cases} \beta \ , & 0\le t\le c_1 \\ 0 \ , & c_1<t\le t_1. \end{cases} \quad (50)$$

where C_1 may or may not be equal to t_1. Now, (48) follows from (50) and the fact that $x_4(0)=1$ (See Fig. 2). ■

III. Avoiding Oblique Shock Waves

In this section we assume the oblique shock waves in the active medium are significant. Thus, we avoid sharp corners as well as high curvatures. For this reason, we assume x_{m+2} is a piecewise C^2 function, and define a new component $x_{m+3}=\dot{x}_{m+2}$. The derivative $\dot{x}_{m+3}$ is taken as the control u which is piecewise continuous and, without loss of generality, left continuous. Hence, if x_{m+2} is assumed to be concave downward with $-\alpha$ as a lower bound for its second derivative, then

$$u(t)\in[-\alpha, 0], \quad x(t)\in \mathbb{R}^{m+1}, \quad (51)$$

where, as we mentioned before, the positive constant α is discussed in [6, pp. 427-430].

In view of (7)-(9) and their justification (27)-(30), the equation of motion in this case is as follows:

$$\dot{x}_i=\frac{1}{x_{m+2}}F_i(x_1,\ldots,x_{m+1}), i=1,2,\ldots,m, \quad (52)$$

$$\dot{x}_{m+1}=\frac{1}{x_{m+2}}[x_{m+3}F_{m+1}(x_1,\ldots,x_{m+1})+F_{m+2}(x_1,\ldots,x_{m+1})], \quad (53)$$

$$\dot{x}_{m+2}=x_{m+3}, \quad (54)$$

$$\dot{x}_{m+3}=u \quad (55)$$

The analogues of (10)-(11) are

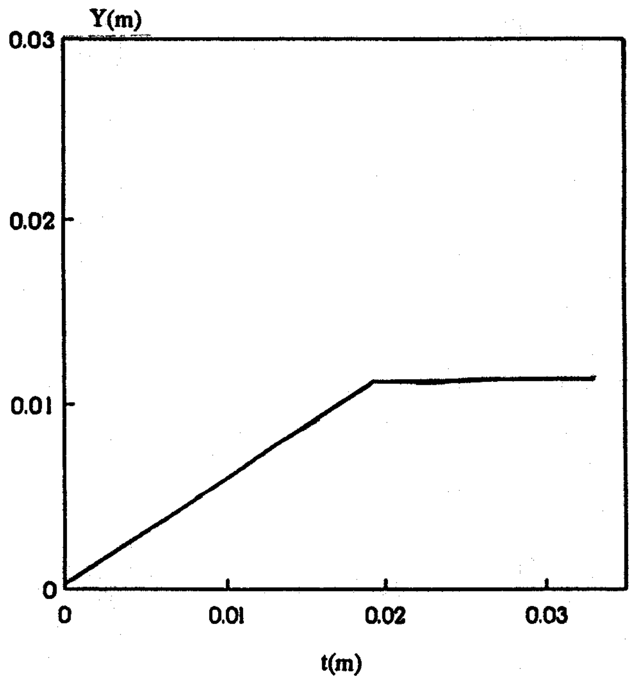


Figure 2. Upper half of an optimal nozzle of piecewise C^1 shape

$$\dot{x}=f(x,u),\ x\in \mathbb{R}^{m+3},\ u\in[-\alpha,0], \tag{56}$$

ınd

$$e=(t_0,t_1,x(t_0),x(t_1))\in \mathbb{R}^{2m+8}. \tag{57}$$

The performance index ϕ_1 and the functions $\phi_2,\ldots,\phi_{m+4}$ ıre now defined on $\mathbb{R}^{2m+8}$ with no other change in (12)-(15). The function ϕ may have additional components lepending on which problem (B1-B4) is under consideration. In the following we reformulate problems B1-B4 to suit our approach.

III.1. Problem. Let $\mathcal{F}$ be the set of all $(m+4)$-tuples $(x,u)=(x_1,\ldots,x_{m+3},u)$ such that $x_1,\ldots,x_{m+3}$ are piecewise C^1 functions and $u:[t_0,t_1]\to[-\alpha,0]$ is a piecewise continuous left continuous function satisfying the equation of motion (56) with the initial conditions (13)-(15). Define $J:\mathcal{F}\to \mathbb{R}$ by $J(x,u)=\phi_1(e)$, where e is as in (57). Find $(x,u)\in\mathcal{F}$ such that J is minimum.

As in Problem II.1, x and u are called the optimal trajectory and the optimal control, respectively. Problem III.1 is a reformulation of Problem B1. Problems B2-B4 will be reformulated respectively as follows.

III.2. Problem. As in Problem III.1 with the extra initial condition

$$\phi_{m+5}(e):=x_{m+3}(t_0)-\beta=0. \tag{58}$$

III.3. Problem. As in Problem III.1 with the extra end condition

$$\phi_{m+5}(e):=x_{m+3}(t_1)=0. \tag{59}$$

III.4. Problem. As in Problem III.1 with the extra end conditions

$$\phi_{m+5}(e):=x_{m+3}(t_0)-\beta=0. \tag{60}$$
$$\phi_{m+6}(e):=x_{m+3}(t_1)=0. \tag{61}$$

A partial solution to Problems III.1-III.4 is given by the following theorem.

III.5. Theorem. Let (x,u) be an optimal solution of Problem III.1, III.2, III.3, or III.4. Then there exist $\lambda_1\in\{-1,0\}$ and $P:[0,t_1]\to\mathbb{R}^{m+3}$ for some $t_1>0$ such that

$$\dot{P}_i=-\sum_{j=1}^{m+3}P_j(t)\,\partial f_j(x(t),u(t))/\partial x_i\ (i=1,\ldots,m+3), \tag{62}$$

$$P_i(t_1)=-\lambda_1\partial g_0(x_1(t_1),\ldots,x_{m+1}(t_1))/\partial x_i\ (i=1,\ldots,m+1), \tag{63}$$

$$P_{m+2}(t_1)=0, \tag{64}$$

$$H(t,x(t),u(t))=0,\quad 0\le t\le t_1, \tag{65}$$

$$u(t)=0 \text{ if } P_{m+3}(t)>0, \tag{66}$$

$$u(t)=-\alpha \text{ if } P_{m+3}(t)<0, \tag{67}$$

where

$$H(t,x,u)=\sum P_j(t)f_j(x,u). \tag{68}$$

Moreover, if the set

$$\{t\in[0,t_1]:P_{m+3}(t)=0,\ u(t)\notin\{0,\alpha\}\} \tag{69}$$

has no interior point, then x_{m+2} consists of finitely many line or parabolic segments.

Finally,

$$P_{m+3}(0)=0 \text{ for Problems III.1 and III.3}, \tag{70}$$

$$P_{m+3}(t_1)=0 \text{ for Problems III.1 and III.2}. \tag{71}$$

Proof. Let $e\in\mathbb{R}^{2m+8}$ be as in (57), and let $\phi:\mathbb{R}^{2m+8}\to\mathbb{R}^s$ be a function whose first $m+4$ components $\phi_1,\ldots,\phi_{m+4}$ are as in (12)-(15). For Problem III.1, $s=m+4$ and hence ϕ is well-determined. For Problems III.2 and III.3, $s=m+5$ and ϕ_{m+5} is defined as in (58) and (59) accordingly. Finally, $s=m+6$ for Problem III.4, and ϕ_{m+5} and ϕ_{m+6} are defined by (60)-(61). It now follows from [4, pp. 27-28] that there exist a nonzero vector $\lambda\in\mathbb{R}^s$ with $\lambda_1\in\{0,-1\}$ and a vector function $P:[0,t_1]\to\mathbb{R}^{m+3}$ satisfying (62) and the following extra conditions:

$$P(t_1)'=\lambda'\phi_{x(1)} \tag{72}$$

$$=\lambda'\begin{bmatrix} -\frac{\partial g_0}{\partial x_1}(x_1(t_1),\ldots,x_{m+1}(t_1))\ldots-\frac{\partial g_0}{\partial x_{m+1}}(x_1(t_1),\ldots,x_{m+1}(t_1)) & 0\\ 0 & \\ A & \end{bmatrix}$$

$$H(t_1,x(t_1),u(t_1))=-\lambda'\phi_{t_1}=0, \tag{73}$$

$$H(t,x(t),u(t))=\lambda_2, \tag{74}$$

$$\max\{H(t,x(t),u):-\alpha\le u\le 0\}=H(t,x(t),u(t)), \tag{75}$$

for all $t\in[0,t_1]$, where A is a vacuous matrix for Problems III.1-III.2, and

$$A=[0\ \ 0\ \ \ldots\ \ 0\ \ 1]_{1\times(m+3)}$$

for Problems III.3-III.4. Now, (63)-(65) follow from (72)-(74). The assertions (66)-(67) follow from (75) and hence, if the set in (69) has an empty interior, then x_{m+2} is piecewise linear or parabolic.

Now, if A is vacuous, then $P_{m+3}(t_1)=0$. The result for $P_{m+3}(0)$ follows from $P(t_0)'=\lambda'\phi_{x(t_0)}$ [4, p. 27, formula (5.4)]. Since the last column of $\phi_{x(t_0)}$ in Problems III.1 and III.3 is zero, it follows that $P_{m+3}(0)=0$.■

III.6. Remark. With the notation of Theorem III.5 and its proof we can further conclude that $\lambda_2=0$ and

$$P_i(0)=\lambda_{i+2}(i=1,\dots,m+2) \quad (76)$$

$$P_{m+3}(0)=\lambda_{m+5} \text{ for Problems III.2 and III.4,} \quad (77)$$

$$P_{m+3}(t_1)=\lambda_{m+5} \text{ for Problem III.3,} \quad (78)$$

$$P_{m+3}(t_1)=\lambda_{m+6} \text{ for Problem III.4.} \quad (79)$$

For more detailed results in this case we again approximate the Anderson's bimodal model by assuming the gas has a frozen nozzle flow.

Here, since $u=\dot{x}_{m+2}=\dot{x}_{m+3}$ it follows from (27)-(32) that the equation of motion is described by the system

$$\dot{x}_i=F_i(x_i,x_3), i=1,2, \quad (80)$$
$$\dot{x}_3=x_5F_3(x_3), \quad (81)$$
$$\dot{x}_4=x_5, \quad (82)$$
$$\dot{x}_5=u. \quad (83)$$

By Theorem III.5, for the optimal solution (x, u) of Problems III.1-III.4, there exists a function P: $[0, t_1]\to \mathbb{R}^5$ such that

$$\dot{P}_i=-P_i\,\partial F_i/\partial x_i (i=1,2), \quad (84)$$
$$\dot{P}_3=-P_1\partial F_1/\partial x_3-P_2\partial F_2/\partial x_3-P_3x_5\,dF_3/dx_3 \quad (85)$$
$$\dot{P}_4\equiv 0, \dot{P}_5=-P_3F_3-P_4, \quad (86)$$
$$H(t,x(t),u(t))=P_1F_1+P_2F_2+P_3x_5F_3+P_4x_5+P_5u\equiv 0. \quad (87)$$

Since $P_4(t_1)=0$, it follows that $P_4\equiv 0$ and hence

$$\dot{P}_5+P_3F_3=P_1F_1+P_2F_2+P_3x_5F_3+P_5u\equiv 0. \quad (88)$$

Now, (39)-(47) hold and hence Lemma II.3 remains valid in this case too. (Note that $\dot{x}_4$ is now continuous and thus $\dot{x}_4(t-)=\dot{x}_4(t+)=\dot{x}(t)$ for all $t\in[0,t_1]$.)

The analogue of Theorem II.4 is as follows.

III.7. Theorem. Assume (x, u) is an optimal solution of Problem III.1, III.2, III.3, or III.4 in which the equation of motion is changed to (80)-(83). Then, allowing degenerate segments, x_4 consists of a line segment, a parabolic segment, and a line segment in the same order. Moreover, if the parabolic segment degenerates to a point, then the supersonic part of the nozzle is a wedge.

Proof. We first show that Problems III.1-III.4 are not abnormal; i.e., $\lambda_1=-1$, where $\lambda=(\lambda_1,\dots,\lambda_s)$ is as in the proof of Theorem III.5. Assume, if possible, that $\lambda_1=0$. Then $P_1\equiv P_2\equiv 0$ and $P_3(t_1)=0$. (See (40) and (63).) It follows from (85) that

$$P_3(t)=P_3(t_1)\exp\int_t^{t_1} x_5(t)\frac{dF_3}{dx_3}(x_3(t))x_3(t)\,dt. \quad (89)$$

Hence $P_3\equiv 0$. By (88), $P_5\equiv P_5(0)=P_5(t_1)$ and $P_5(0)u\equiv 0$. By (70)-(71), $P_5\equiv 0$ for Problems III.1-III.3. Also, since $x_5(0)=\beta>0=x_5(t_1)$ and (hence) $u\neq 0$ in Problem III.4, it follows that $P_5\equiv 0$ for that problem. Thus, in view of Remark III.6, $\lambda=0$; a contradiction. Hence $\lambda_1=-1$.

Next, we show that the set (69) has an empty interior. If $P_5(t)=0$ and $u(t)<0$, for all t in some interval (ξ,ζ), then, in view of (88) and the fact that $F_3<0$, $P_3(t)=P_1(t)F_1(x_1(t),x_3(t))+P_2(t)F_2(x_2(t),x_3(t))=0$ and $\dot{x}_4(t)=x_5(t)>0$ for all $t\in(\xi,\zeta)$ which is impossible in view of Lemma II.3. Thus, $u(t)\in\{0,-\alpha\}$ for all $t\in[0,t_1]$. Thus, there exists a partition $\{0=C_0<\dots<C_k=t_1\}$ such that u has a constant value $-\alpha$ or 0 on each subinterval $(C_{i-1}, C_i]$ $(i=1,\dots,k)$. Moreover, if $k>1$, then u is discontinuous at $C_i (i=1,\dots,k-1)$. We claim $k\leq 3$. In view of (88),

$$P_5(t)=x_5(t)\left[\int_0^t \frac{P_1F_1+P_2F_2}{x_5^2}dt+\frac{P_5(0)}{x_5(0)}\right]=x_5(t)\,\pi(t), \quad (90)$$

whenever $x_5(t)\neq 0$. Let $(0,t_2)$ be the largest open interval on which $x_5>0$. Since the nozzle $x_4=$ constant is physically rejected, either $t_2=C_{k-1}$ or $t_2=C_k=t_1$. Since $x_5(t)>0$ for all $t\in[0,t_2)$, it follows that $\pi(t)P_5(t)$ have the same sign for all $t\in[0,t_2)$. Now,

$$x_5^2(t)\,\dot{\pi}(t)=P_1(t)F_1(x_1(t),x_3(t))+P_2(t)F_2(x_2(t),x_3(t)) \quad (91)$$

for all $t\in[0,t_2)$. By Lemma II.3, there exists some $t_3\in[0,t_2]$ such that $\dot{\pi}(t)<0$ if $0<t<t_3$ and $\dot{\pi}(t)>0$ if $t_3<t<t_2$. (Note that either of the intervals $(0,t_3)$ or (t_3,t_2) may be empty.) We consider two cases.

Case 1. $P_5(0)\leq 0$. Then $\pi(0)\leq 0$ and hence there exists

$\in [0, t_2]$ such that $\pi(t)<0<\pi(s)$ whenever $0<t<t_4<s<t_2$. ius, $u(t)=-\alpha$ for $t \in (0, t_4)$ and $u(t)=0$ for $t \in (t_4, t_2)$. Since is continuous, it follows that either $t_4=t_2$ or $t_2=t_1$. Hence ≤ 2.

Case 2. $P_5(0)>0$. Then $\pi(0)>0$ and hence there exists and t_5 such that $0 \leq t_4 \leq t_5 \leq t_2$, $\pi(t)>0$ if $t \in (0, t_4) \cup (t_5, t_2)$, ıd $\pi(t)<0$ if $t \in (t_4, t_5)$. Accordingly, $u=0$ on $(0, t_4) \cup (t_5,$), and $u=-\alpha$ on (t_4, t_5). Again, here, since x_5 is continuous, follows that either $t_5=t_2$ or $t_2=t_1$. Thus $k \leq 3$.

So, to sum up, we have shown that x_4 consists of a line :gment, a parabolic segment, and a line segment in the ıme order (See Fig. 3), where we allow some of them to :generate to a point. Note that two line segments can :ver be adjacent.■

III.8. Conclusion

In this section, we prove that the supersonic part of an ɔtimal gas dynamic laser with a gas mixture of CO_2 $00X_{co_2}\%)$, N_2 $(100X_{N_2}\%)$, and H_2O is bounded by the lanes $t=0$, $t=t_1$, and $z=\pm \omega/2$, and the cylinders $y=hx_4(t)/$ described as

$$y=Ct+h/2 \quad \text{for} \quad 0 \leq t \leq D,$$

$$=-(\alpha/2)t^2+(C+\alpha D)t+(h-\alpha D^2)/2 \ \text{for}\ D \leq t \leq (C+\alpha D-E)/\alpha,$$
$$=Et+h/2+(C+\alpha D-E)^2/(2\alpha)-\alpha D^2/2,\ \text{for}\ (C+\alpha D-E)/\alpha \leq t$$
$$t_1,$$

'here h, ω are the height and the width of the throat, C is ıe slope of the opening wedge, D is the length of the 'edge part, and E is the slope of the nozzle at its end $t=t_1$.

Now, assuming the combustion chamber has a known esign and subject to the reductions and models imposed y [1,6,8,9,10,12,13,14], the parameters X_{CO_2}, X_{N_2}, C, D, ', t_1 together with the gas temperature T_0 and pressure P_0 ı the combustion chamber fully determine the optimal hape of our laser via the system of differential equations 7)-(9), exactly in the same way done by Losev-Makarov 3,9,10] and Reddy-Shanmugasundaram [13,14]. (Folɔwing [6, pp. 427-430], we take $\alpha=20$). Thus, our optimal ontrol problem reduces to a multifactor optimal problem. 'his is an improvement on the work done by Losev-Aakarov [8,9,10]; now, we already know that the superonic part of the nozzle begins with a wedge, then bends to parabolic cylinder and ends with a second wedge.

Our numerical computations reveal that E must be 0, nd $2\tan^{-1}C$ must be the maximum admissible opening of he wedge at the throat which is usually 40° [7]. In Figure ·, the values of the small signal gain g_0 is plotted gainst the final half-angle $A=\tan^{-1}E$ for various values f $B=\tan^{-1}C$ and fixed values of the gas temperature and ressure in the combustion chamber.

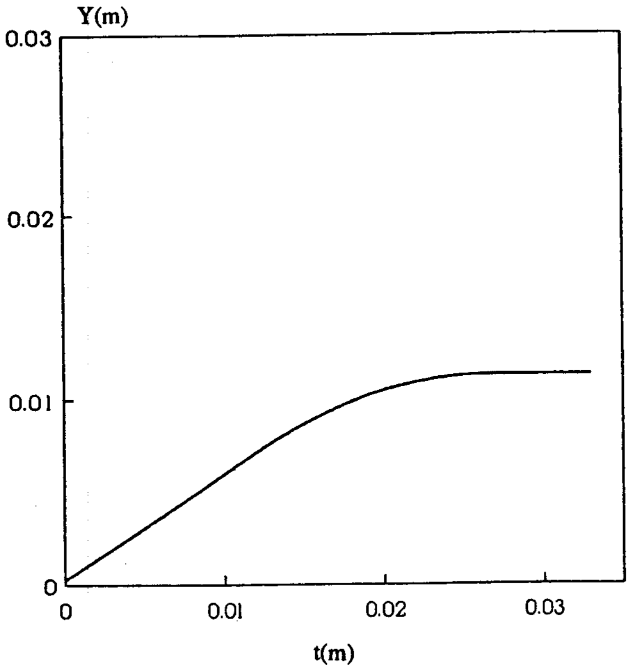


Figure 3. Upper half of an optimal nozzle of piecewise C^2 shape

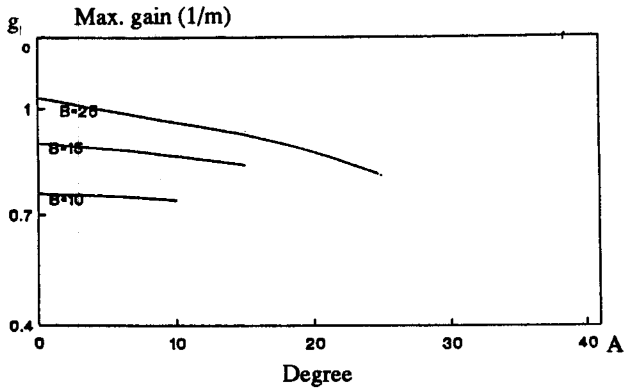


Figure 4. Gain versus beginning and ending angles of a piecewise C^2 nozzle shape T_0=1200°K, P_0=40 Atm, X_{CO_2}= 0.1, X_{N_2}= 0.85, X_{H_2O}= 0.05

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