IRRELEVANT ATTACHED PRIME IDEALS OF A CERTAIN ARTINIAN MODULE OVER A COMMUTATIVE RING

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Abstract

Let M be an Artinian module over the commutative ring A (with nonzero identity) and $\mathbf{a} \subseteq \mathbf{p} \in$ spec A be such that $\mathbf{a}$ is a finitely generated ideal of A and $\mathbf{a}M = M$. Also suppose that $H = \oplus_{i \in \mathcal{Z}} H_i$ where $H_i = M/(0:_M \mathbf{a}^{-i})$ for $i<0$ and $H_i = M$ for $i \geq 0$; and $\mathcal{R} = A[\mathbf{a}T, T^{-1}]$ is the Rees ring of A with respect to $\mathbf{a}$ (T is an indeterminate). In [12] it is shown that H is an $\mathcal{R}$-module. In this paper, we give various conditions under which the prime ideal $(\mathbf{p}, \mathbf{a}T, T^{-1})\mathcal{R}$ is an attached prime ideal of $(0:_H \mathcal{R}T^{-1})$ as an $\mathcal{R}$-module.

Introduction

In [5], the concept of the relevant component of an ideal I (denoted by I*) of a Notherian ring R was introduced; moreover, the arguments in [5, 8] prove that I* is an interesting and useful ideal. S. McAdam [3] discussed the conditions under which $(\mathbf{p}, IT, T^{-1})S$ is a prime divisor of ST^{-1} in the Rees ring $S = R[IT, T^{-1}]$ of R with respect to I, and for establishing this, he used the nice properties of I*. ($I \subseteq \mathbf{p} \in$ spec R).

The present author [11], defined and developed a satisfactory concept of the relevant component of an ideal $\mathbf{a}$ of a commutative ring A (with nonzero identity) relative to an Artinian A-module. It is appropriate for us to begin by summarizing some of the main points.

Let A be a commutative ring (with nonzero identity), M an Artinian A-module and $\mathbf{a}$ an ideal of A. The relevant component of $\mathbf{a}$ (relative to M), denoted by $\mathbf{a}^*$ is defined as $\mathbf{a}^* = \mathrm{ann} \bigcap_{i \geq 1} \mathbf{a}^i (0:_M \mathbf{a}^{i+1})$. Then, from the Artinian property of M it follows that for some large enough k,

$\mathbf{a}^* = \mathrm{ann}_A(\mathbf{a}^k(0:_M \mathbf{a}^{k+1}) = ((0:_M \mathbf{a}^k) :_A ((0:_M \mathbf{a}^{k+1})$. Moreover, if $\mathbf{a}$ is such that $\mathbf{a}M = M$, then [11, 2.2], for large n, $(0:_M (\mathbf{a}^*)^n) = (0:_M \mathbf{a}^n)$ and $\mathbf{a}^*$ is the largest ideal with this property.

Again let M be an Artinian A-module and $\mathbf{a}$ an ideal of A such that $\mathbf{a}M = M$. In this paper, using the Artinian property of M, we explore more interesting results concerning the relevant component of $\mathbf{a}$ relative to M. Moreover, we establish results which are, in a sense, dual to those of [3].

Throughout the paper, A will denote a commutative ring (with nonzero identity), M will denote an Artinian A-module and $\mathbf{a}$ will be an ideal of A. We use $\mathcal{Z}$ to denote the set of integers and IN to denote the set of positive integers. The integral closure of $\mathbf{a}$ relative to M will be denoted by $\bar{\mathbf{a}}$. (For definition of integral closure see [10]).

Keywords: Artinian modules; Relevant component

Some Preliminary Results

In this section, we shall prove some results which will be needed later.

Lemma 2.1. Let $\mathbf{a}$ be an ideal of A such that $\mathbf{a}M = M$. Then:

(i) $(\mathbf{a}^n)^* = \mathrm{ann}_A (\bigcap_{i \geq 1} \mathbf{a}^i (0:_M \mathbf{a}^{i+n}))$.

(ii) $\mathbf{a}^* \supseteq (\mathbf{a}^2)^* \supseteq \ldots \supseteq (\mathbf{a}^n)^*$.

(iii) $\mathbf{a} \subseteq \mathbf{a}^* \subseteq \bar{\mathbf{a}}$.

(iv) $\mathbf{a}(\mathbf{a}^n)^* \subseteq (\mathbf{a}^{n+1})^*$.

(v) If $\mathbf{a} \subseteq \mathbf{b}$, then, $\mathbf{b} \subseteq \mathbf{a}^*$ if and only if, there is $t \geq 1$ such that $(0:_M \mathbf{b}^t) = (0:_M \mathbf{a}^t)$.

Proof. (i) By definition of $\mathbf{a}^*$ we have, for large k,

$(\mathbf{a}^n)^* = \mathrm{ann}_A (\bigcap_{i\geq 1} \mathbf{a}^{ni} (0:_M \mathbf{a}^{ni+n})) = \mathrm{ann}_A (\mathbf{a}^{nk} (0:_M \mathbf{a}^{nk+n}))$

$= \mathrm{ann} (\bigcap_{i\geq 1} \mathbf{a}^i (0:_M \mathbf{a}^{i+n}))$

(ii) By (i), we have, for $t \in \mathrm{IN}$,

$(\mathbf{a}^t)^* = \mathrm{ann} (\bigcap_{i\geq 1} \mathbf{a}^i (0:_M \mathbf{a}^{i+t})) \supseteq \mathrm{ann}_A (\bigcap_{i\geq 1} \mathbf{a}^i (0:_M \mathbf{a}^{i+t+1}))$
$= (\mathbf{a}^{t+1})^*$.

(iii) Result follows from the definition of $\mathbf{a}^*$ and [10, (2.4) (i)].

(iv) It follows from (i) and the minimal condition that for all large enough k

$(\mathbf{a}^n)^* = (0:_A \mathbf{a}^k (0:_M \mathbf{a}^{k+n}))$. Thus $\mathbf{a} (\mathbf{a}^n)^* = \mathbf{a} (0:_A \mathbf{a}^k (0:_M \mathbf{a}^{k+n}))$
$\subseteq (0:_A \mathbf{a}^k (0:_M \mathbf{a}^{k+n+1}))$.
$= (\mathbf{a}^{n+1})^*$, by (i).

(v) Let $\mathbf{a} \subseteq \mathbf{b}$ and, for some $t\geq 1$, $(0:_M \mathbf{b}^t) = (0:_M \mathbf{a}^t)$. Then $\mathbf{b} (0:_M \mathbf{a}^{t+1}) = \mathbf{b} ((0: \mathbf{a}^t):_M \mathbf{a}) = \mathbf{b} ((0:_M \mathbf{b}^t):_M \mathbf{a}) \subseteq (\mathbf{b} (0:_M \mathbf{b}^t):_M \mathbf{a}) \subseteq (0:_m \mathbf{b}^{t-1}):_M \mathbf{a}) \subseteq (0:_A \mathbf{a}^t)$. Therefore $\mathbf{b} \subseteq ((0:_M \mathbf{a}^t):_A (0:_M \mathbf{a}^{t+1})) \subseteq \mathrm{ann}_A (\bigcap_{i\geq 1} \mathbf{a}^i (0:_M \mathbf{a}^t)) = \mathbf{a}^*$.

Conversely, let $\mathbf{b} \subseteq \mathbf{a}^*$. Then by [11, 2.2], for all large t,

$(0:_M \mathbf{a}^t) \supseteq (0:_M \mathbf{b}^t) \supseteq (0:_M (\mathbf{a}^*)^t) = (0:_M \mathbf{a}^t)$.

Thus $(0:_M \mathbf{b}^t) = (0:_M \mathbf{a}^t)$ for large enough t.

For the next result, suppose that $\mathbf{a} = (a_1, \ldots, a_s)$ and T is an indeterminate. Further suppose that $\mathcal{R} := A[a_1T \ldots, a_sT, T^{-1}]$ ($R = A[a_1T, \ldots, a_sT]$ is the Rees ring (restricted Rees ring) of A with respect to $\mathbf{a}$ and graded in the usual way by $\mathbb{Z}$.

Lemma 2.2. (See [12, (3.10)] With the same notation as above, set $H = \bigoplus_{x\in} \mathcal{H}_n$, where for $n \in \mathbb{Z}$,

$$H_n = \begin{cases} M & \text{if } n \geq 0, \\ M/(0:_M \mathbf{a}^{-n}) & \text{if } n < 0. \end{cases}$$

Then H has a structure as an $\mathcal{R}$- module. Further, put $G = (0:_H \mathcal{R} T^{-1})$. Then $G = \bigoplus_{x\in\mathbb{Z}} G_n$ where, for $n \in \mathbb{Z}$,

$$G_n = \begin{cases} 0 & \text{if } n > 0, \\ (0:_M \mathbf{a}^{-n+1})/(0:_M \mathbf{a}^{-n}) & \text{if } n \leq 0 \end{cases}$$

and G is an Artinian $\mathcal{R}$- module.

Proof. We turn H into an $\mathcal{R}$- module as follows:

$a_i T (x + (0:_M \mathbf{a}^{-n})) = a_i x + (0:_M \mathbf{a}^{-n+1})$ and,

$T^{-1} (x + (0:_M \mathbf{a}^{-n})) = x + (0:_M \mathbf{a}^{-n+1})$,

where $x \in M$, $1 \leq i \leq s$ and $n \in \mathbb{Z}$. It is easy to see that H is a module over $\mathcal{R}$.

For the last part, we note that G is an $\mathcal{R}$-submodule of H. Also by [13, 2.2] G is an Artinian R-module and obviously an Artinian $\mathcal{R}$-module.

Before stating the next Lemma, we recall that there is a theory of secondary representation for Artinian A-modules which is in many respects dual to primary decomposition for Noetherian A-modules. Accounts of this theory are available in [2,7]; however, we shall use the terminology of [2] about this theory. This theory associates with the Artinian A-module M a finite collection of prime ideals called the attached prime ideals of M, denoted by $\mathrm{Att}_A(M)$ (or Att (M)). (It is convenient to take the view that the zero A-module is the sum of the empty family of its secondary submodules.)

In [9], Sharp proved that sets of sequences $\{\mathrm{Att}(0:_M \mathbf{a}^n)\}$, $n \in \mathrm{IN}$, and $\{\mathrm{Att}(0:_M \mathbf{a}^{n+1}) / (0:_M \mathbf{a}^n)\} n \in \mathrm{IN}$ are ultimately constant. We denote these eventual stable values by $At^*(\mathbf{a}, M)$ and $Bt^*(\mathbf{a}, M)$, respectively.

Lemma 2.3. Let N be an A-module and $x \in A$ be such that $xN = N$ and $(0:_N x)$ be an Artinian A-module. Then for all $n \in \mathrm{IN}$, $(0:_N Ax^n)$ is an Artinian A-module, and for all n, $\mathrm{Att}_A (0:_N Ax^n) = \mathrm{Att}_A (0:_N Ax)$.

Proof. It is easy to see that the submodule $(0:_N Ax)$ is isomorphic to the quotient module $(0:_N Ax^t) / (0:_N Ax^{t-1})$ for all $t\geq 1$. Next the sequence.

$$0 \to (0:_N Ax) \to (0:_N Ax^2) \to (0:_N Ax^2)/ (0:_N Ax) \to 0$$

is exact. Thus $(0:_N Ax^2)$ is an Artinian A-module and by [2, 4.1]

$$\begin{aligned} \mathrm{Att}(0:_N Ax) &= \mathrm{Att}\,[(0:_N Ax^2)/ (0:_N Ax)] \subseteq \mathrm{Att}\,(0:_N Ax^2) \\ &\subseteq \mathrm{Att}\,[(0:_N Ax^2)/ (0:_N Ax)] \cup \mathrm{Att}\,(0:_N Ax) \\ &= \mathrm{Att}\,(0:_N Ax). \end{aligned}$$

So $\mathrm{Att}\,(0:_N Ax^2) = \mathrm{Att}\,(0:_N Ax)$. The result follows by induction on n.

The Main Results

Throughout this section, $\mathbf{a}$ is an ideal of A such that $\mathbf{a}M=M$. First of all we need a Lemma which is given below.

Lemma 3.1. With the same notation and assumptions as in 2.2, let $\mathbf{p} \supseteq \mathbf{a}$ with $\mathbf{p} \in \mathrm{spec}\,A$. Let $\mathbf{q} := \mathbf{p} + \mathbf{a}T + \mathbf{a}^2T^2 + \ldots$

and $P: AT^{-1}+p+aT+a^2T^2+\ldots$ Then $q \in Att_R(G)$ if and only if $P \in Att_{\mathscr{R}}(G)$.

Proof. Let $q \in Att_{\mathscr{R}}(G)$. Then there is an R-quotient of G, say N, such that $q = \sqrt{ann_R N}$. Since G is an $\mathscr{R}$-module in an obvious way, N is an $\mathscr{R}$-quotient of G as well and this structure is such that $T^{-1}N = 0$, i. e. $T^{-1} \in ann_{\mathscr{R}} N$. Now it is easy to see that $\sqrt{ann_{\mathscr{R}} N} \cap A = \sqrt{ann_R N} \cap A = q \cap A = p$ and $\sqrt{ann_{\mathscr{R}} N} \supseteq \sqrt{ann_R N} = q$. Thus $\sqrt{ann_{\mathscr{R}} N} = (p, aT, T^{-1}) = P$.

Conversely, let $P \in Att_{\mathscr{R}} G$. Then again there is an $\mathscr{R}$-quotient module of G such that $P = \sqrt{ann_{\mathscr{R}} N}$. Now the result follows from the relations

$$q = P \cap R = \sqrt{ann_{\mathscr{R}} N} \cap R = \sqrt{ann_R N}$$

Theorem 3.2. Let the notation be the same as in 2.2 and $p \supseteq a$ with $p \in spec(A)$ be ideals of A. Then the following are equivalent:

(i) $p \in Att_A[(0:_M a^n)/(0:_M (a^n)^*)]$ for some $n \geq 1$;

(ii) for some $n \geq 1$, there is an ideal $b_n \supseteq a^n$ such that $p \in Att[(0:_M a^n)/(0:_M b_n)] \setminus Att[(0:_M a^{n+1})/(0: ab_n)]$;

(iii) $P = \ldots + AT^{-1} + p + aT + a^2T^2\ldots$ belongs to $Att_{\mathscr{R}}(0:_H RT^{-1})$;

(iv) $p \in Att_A[(0:_M a^n)/a^k(0:_M a^{n+k})]$ for some $n \geq 1$ and large k.

Proof. By [1, Lemma 3], we can (and do) assume that **a** is finitely generated.

(i)$\Rightarrow$(ii) By [11, 2.2], for all large enough k, $(0:_M a^k) = (0:_M (a^k)^*)$. Thus, suppose that n is chosen so that $p \in Att[(0:a^n)/(0:(a^n)^*]$ and $p \notin Att[(0:_M a^{n+1})/(0:(_M a^{n+1})^*)]$.

By Lemma 2.1, $(0:_M a(a^n)^*) \supseteq (0:_M (a^{n+1})^*)$ so that the sequence

$0 \to (0:_M a(a^n)^*/(0:_M (a^{n+1})^*) \to (0:_M a^{n+1})/(0:_M (a^{n+1})^*) \to (0:_M a^{n+1})/(0:_M a(a^n)^*) \to 0$ is exact. Thus by [2, 4.1] $Att_A[(0:_M a^{n+1})/(0:_M a(a^n)^*)] \subseteq Att[(0:_M a^{n+1})/(0:_M (a^{n+1})^*]$ and therefore $p \in Att(0:_M a^n)/(0:_M (a^n)^*)] \setminus Att[(0:a^{n+1})/(0:a(a^n)^*)]$. So we put $b_n = (a^n)^*$.

(ii)$\Rightarrow$(iii) Let $p \in Att[(0:_M a^n)/(0:_M b_n)] \setminus Att[(0:_M a^n)/(0:_M ab_n)$ with $b_n \supseteq a^n$. Suppose that $p = \sqrt{0:_A S_1}$ where S_1 is a quotient of $(0:_M a^n)/(0:_M b_n)$, say, $S_1 \approx (0:_M a^n)/N$ with $(0:_M a^n) \supseteq N \supseteq (0:_M ab_n)$. We claim that, $q = p + aT + a^2T^2 + \ldots \in Att_R(0:_H \mathscr{R}T^{-n})$ and the result follows from Lemmas 2.3 and 3.1. To see this, we note that $(0:_H \mathscr{R}T^{-n}) = \bigoplus_{x \in \mathbb{Z}} H_i$ where for $i \in \mathbb{Z}$,

$$H_i = \begin{cases} (0:_M a^{-i+n})/(0:_M a^{-i}) & \text{if } i<0 \\ (0:_M a^{n-i}) & \text{if } i \geq 0 \end{cases}$$

(with the convention that $a^n = A$ for $n \leq 0$). We shall prove that **q** is minimal over $\sqrt{0:_R N'}$ with $N' = \bigoplus_{x \in \mathbb{Z}} N'_i$, where for $i \in \mathbb{Z}$,

$$N'_i = \begin{cases} (0:_M a^{n-i})/(N:_M a^{-i}) & \text{if } i \leq 0, \\ 0 & \text{if } i>0. \end{cases}$$

To see this, let $\alpha \in ann_R(N') \cap A$. Then $\alpha(0:_M a^n) \subseteq N$ and so $\alpha \in (0:_A S_1) \subseteq p$. Now let $\alpha \in p$. Then $\alpha^t(0:_M a^n) \subseteq N$ for some $t \in \mathbb{N}$ and so $\alpha^t(0:_M a^{n-i}) \subseteq (N:_M a^{-i})$ for all $i \leq 0$. Thus, $\alpha \in ann_R(N)' \cap A$. Therefore $ann_R(N') \cap A = p$

Now suppose that the claim is false and **q'** is a prime ideal of R minimal over $ann_R N'$ such that $q' \subset q$. Then, by the above argument, we have

$$p = ann_R(N') \cap A \subseteq q' \cap A \subseteq q \cap A = p$$

so $q' \cap A = p$. Now since $p \subseteq q$, we must have a $T \not\subset q$. Let $\alpha \in a$ be such that $\alpha T \notin q$. Then

$((0:N'):_A \alpha TR) \subseteq (q:_A \alpha TR) = q$ and

$((N:_M \alpha):_A (0:_M a^{n+1})) = ((0:N'):_A \alpha TR) \cap A \subseteq q \cap A = p$.

Also $(0:_A S_1) \subseteq ((N:_M a):_A (0:_M a^{n+1})) \subseteq ((N:\alpha):_A (0:_M a^{n+1}))$. Thus

$$p = \sqrt{0_A : S_1} \subseteq \sqrt{((N:_M a):_A (0:_M a^{n+1}))} \subseteq p \text{ and } p = \sqrt{(N:_M a):_A (0:_M a^{n+1})}.$$

We deduce that $p \in Att[(0:_M a^{n+1})/(0:_M ab_n)]$, a contradiction.

(iii)$\Rightarrow$(iv) Let $P \in Att_{\mathscr{R}}(G)$. Then, by Lemma 3.1, $q \in Att_R(G)$. Suppose that $q = \sqrt{0:_R N}$, where N is a quotient of G, say,

$$N = \ldots + (0_M : a^k)/N_k + (0_M a^{k-1})/N_{k-1} + \ldots + (0:_M a)/N_1 + 0 + 0 + \ldots.$$

Thus, for $i \geq 1$, $N_i \subseteq (0:_M a^i)$ and $aN_{i+1} \subseteq N_i$. Since $0 \neq N$ there is $k \geq 1$ such that $N_k \neq (0:_M a^k)$. Let n be the least integer i such that $N_i \neq (0:_M a^i)$. Consider the R-module

$N' = \ldots + (0:_M a^k) / (N_n : a^{k-n}) + \ldots + (0:_M a^{n+1}) / (N_n : a) + (0:_M a^n)/N_n + 0 + 0 + \ldots,$

which is isomorphic to a quotient of N. So N' is **q**-secondary R-module and $\mathbf{q} = \sqrt{0_R : N'}$. But it is easy to see that $\sqrt{0_R : N'} \cap A \subseteq \sqrt{N_n :_A (0_M : \mathbf{a}^n)}$ and from the relation $\mathbf{a}^k (0_M : \mathbf{a}^{k+n}) \subseteq (0_M : \mathbf{a}^n)$ the reverse inclusion will follow, Thus

$$\sqrt{N_n : (0_M : \mathbf{a}^n)} = \sqrt{0_R : N'} \cap A = \mathbf{q} \cap A = \mathbf{p}.$$

Next **a** is finitely generated and hence for all large $k (k \in \mathbb{N})$ $\mathbf{a}^k T^k N' = 0$, i. e. $\mathbf{a}^k (0:_M \mathbf{a}^{n+k}) \subseteq N_n$. Therefore $\mathbf{p} \in \mathrm{Att}\,[(0:_M \mathbf{a}^n)/\mathbf{a}^k (0:_M \mathbf{a}^{n+k})]$.

(iv)$\Rightarrow$(i) Let $(0:_M \mathbf{a}^n)/N$ be a **p**-secondary quotient of $(0:_M \mathbf{a}^n)/\mathbf{a}^k (0:_M \mathbf{a}^{k+n})$ (k is large enough), where $\mathbf{a}^k (0:_M \mathbf{a}^{n+k}) \subseteq N \subseteq (0:_M \mathbf{a}^n)$. By Lemma (2.1) (i), $(\mathbf{a}^n)^* = (0:_A \mathbf{a}^k (0:_M \mathbf{a}^{n+k})) \supseteq (0:_A N)$ for large k. So $(0:_M (\mathbf{a}^n)^*) \subseteq (0:_M (0:_A N)) := N_1$. Now $(0:_M \mathbf{a}^n)/N_1$ is isomorphic to a quotient of $(0_M : \mathbf{a}^n)/N$ and hence **p**- secondary. On the other hand, $(0:_M \mathbf{a}^n)/N_1$ is isomorphic to a quotient of $(0:_M \mathbf{a}^n)/(0:_M (\mathbf{a}^n)^*)$ and so $\mathbf{p} \in \mathrm{Att}\,[(0:_M \mathbf{a}^n)/(0:_M (\mathbf{a}^n)^*)]$.

Corollary 3.3. Let P, **p** and G be the same as in 3.1. Then P will be an attached prime ideal of G (as an $\mathscr{R}$-module) in each of the following cases:

(1) $\mathbf{p} \in \mathrm{Att}\,(0:_M \mathbf{a}^n) \backslash \mathrm{Att}\,(0:_M (\mathbf{a}^n)^*)$ for some $n \geq 1$;

(2) $\mathbf{p} \in \mathrm{Att}\,(0:_M \mathbf{a}^n) \backslash \mathrm{Att}\,(0:_M \mathbf{a}^{n+1})$ for some $n \geq 1$;

(3) $\mathbf{p} \in \mathrm{Att}\,[(0:_M \mathbf{a}^n)/(0:_M \mathbf{a}^{n-1})] \backslash \mathrm{Att}\,[(0:_M \mathbf{a}^{n+1})/(0:_M \mathbf{a}^n)]$ for some $n \geq 1$;

(4) $\mathbf{p} \in \mathrm{Att}\,[(0:_M \mathbf{a}^n)/(0:_M \overline{\mathbf{a}^n})] \backslash \mathrm{Att}\,[(0:_M \mathbf{a}^{n+1})/(0:_M \overline{\mathbf{a}^{n+1}})]$ for some $n \geq 1$.

Proof. From the exact sequence

$$0 \rightarrow (0:_M (\mathbf{a}^n)^*) \rightarrow (0:_M \mathbf{a}^n) \rightarrow (0:_M \mathbf{a}^n)/(0:_M (\mathbf{a}^n)^*) \rightarrow 0$$

and [2, 4.1] we get

$$\mathrm{Att}\,(0:_M \mathbf{a}^n) \subseteq \mathrm{Att}\,(0:_M (\mathbf{a}^n)^*) \cup \mathrm{Att}\,[(0:_M \mathbf{a}^n)/(0:_M (\mathbf{a}^n)^*)].$$

So if **p** is as in (1), then $\mathbf{p} \in \mathrm{Att}\,[(0:_M \mathbf{a}^n)/(0:_M (\mathbf{a}^n)^*]$ for some $n \geq 1$; and hence the result follows from Theorem (3.2)(i) $\Rightarrow$(iii).

(2), (3). For (2), let $\mathbf{b}_n = A$ and for (3) let $\mathbf{b}_n = \mathbf{a}^{n-1}$. Then $\mathbf{p} \in \mathrm{Att}\,[(0:_M \mathbf{a}^n)/(0:_M \mathbf{b}_n)] \backslash \mathrm{Att}\,[(0:_M \mathbf{a}^{n+1})/(0:_M \mathbf{a}\mathbf{b}_n)]$. Since $\mathrm{Att}[(0:_M \mathbf{a}^{n+1})/(0:_M \mathbf{a}\mathbf{b}_n)] \subseteq \mathrm{Att}\,[(0:_M \mathbf{a}^{n+1})/(0:_M \mathbf{b}_n)]$, the result follows from Theorem 3.2 (ii)$\Rightarrow$(iii).

(4). Let $\mathbf{b}_n = \overline{\mathbf{a}^n}$. Then $\mathbf{p} \in \mathrm{Att}[(0:_M \mathbf{a}^n)/(0:_M \mathbf{b}_n)]$ and $\mathbf{p} \notin \mathrm{Att}\,[(0:_M \mathbf{a}^{n+1})/(0:_M \mathbf{b}_{n+1})]$. But by [10, 1.3 (ii)] $\mathbf{a}^{n+1}$ is a reduction of $\mathbf{a}\,\overline{\mathbf{a}^n}$ (relative to M) and hence by [10, 2.4 (i)] $\mathbf{a}\,\overline{\mathbf{a}^n} \subseteq \overline{\mathbf{a}^{n+1}}$. Thus $\mathbf{a}\mathbf{b}_n \subseteq \mathbf{b}_{n+1}$, and so $\mathbf{p} \in \mathrm{Att}\,[(0:_M \mathbf{a}^n)/(0:_M \mathbf{b}_n)] \backslash \mathrm{Att}\,[(0:_M \mathbf{a}^{n+1})/(0:_M \mathbf{a}\mathbf{b}_n)]$. To complete the proof we use Theorem (3.2) (ii)$\Rightarrow$(iii) once more.

For the last Theorem, we need a Lemma, which is given below. This Lemma is essentially Theorem 2.9 of [13] and so we omit the proof.

Lemma 3.4. Let the notation be as in Lemma 2.2 and further suppose that **b** denotes the ideal $\sum_{i=1}^{s} \mathscr{R}(\mathbf{a}_i T)$ of $\mathscr{R}$. Then $\mathbf{p} \in \mathrm{Bt}^*\,(\mathbf{a}, M)$ if and only if there exists $\mathbf{q} \in \mathrm{Att}_{\mathscr{R}}\,(G)$ such that $\mathbf{b} \not\subseteq \mathbf{q}$ and $\mathbf{q} \cap A = \mathbf{p}$.

Theorem 3.5. Let the notation be the same as in Lemma 2.2. Let $n \in \mathbb{N}$ be such that $\mathbf{p} \in \mathrm{Att}\,(0:_M \mathbf{a}^n) \backslash \mathrm{At}^*(\mathbf{a}, M)$. Then P is the only attached prime ideal of G (as an $\mathscr{R}$-module) which intersect A at **p**.

Proof. We may assume (and do so) that $\mathbf{p} \in \mathrm{Att}\,(0:_M \mathbf{a}^n) \backslash \mathrm{Att}\,(0:_M \mathbf{a}^{n+1})$. Then, by Corollary 3.3 (2), $P \in \mathrm{Att}_{\mathscr{R}}\,(G)$. But $\mathbf{p} \in \mathrm{Bt}^*\,(\mathbf{a}, M)$ and hence, by Lemma 3.4, P is the only attached prime ideal of G which intersects A at **p**.

Examples

In this section, we give two examples concerning the ideas we have encountered. The first example is, in fact, an adaptation of Example A of [3] to the Artinian situation.

Example 4.1. Let F be a field and x be an indeterminate. Let $M = F[x^{-1}]$ be the inverse polynomial module. By [1, §2], M is an Artinian F [x]- module. Let $A = \{\alpha_0 + x^3 g(x) \mid g(x)\, F[x]\}$, $\mathbf{p} = xF[x] \cap A$ and $\mathbf{a} = (x^3, x^4)\,A$. Then $\mathbf{a} \neq \mathbf{a}^* = \bar{\mathbf{a}} = \mathbf{p}$. Also, for $n \geq 2$, $\mathbf{a}^n = (\mathbf{a}^n)^* = \overline{\mathbf{a}^n} = \mathbf{p}^n$. Finally, with the same notation as Lemma 2.2 $(\mathbf{p}, \mathbf{a}T, T^{-1})\mathscr{R}$ is not an attached prime ideal of G and hence

$$\mathrm{Att}\,(0:_M \mathbf{a}) \subseteq \mathrm{Att}\,(0:_M \mathbf{a}^2) \subseteq \ldots\ldots.$$

Proof. First we show that M is an Artinian A-module. We note that

$$M = \bigcup_{i=1}^{\infty} (0:_M (xF[x])^i) \subseteq \bigcup_{i=1}^{\infty} (0:_M (xF[x] \cap A)^i) = \bigcup_{i=1}^{\infty} (0:_M \mathbf{p}^i).$$

Thus, $M = \bigcup_{i=1}^{\infty} (0:_M \mathbf{p}^i)$. By [4, 1.3], it is enough to show that $(0:_M \mathbf{p})$ is an Artinian A-module. Now, since A is a Northerian ring and $(0:_M\mathbf{p}) = (0:_M x^3) = \{\alpha_0 + \alpha_1 x^{-1} + \alpha_2 x^{-2} | \alpha_i \in F\}$, $(0:_M\mathbf{p})$ is a Noetherian A-module. On the other hand, $(0:_M\mathbf{p})$ is annihilated by p, which is a maximal ideal of A, ($xF[x]$ is a maximal ideal of $F[x]$, and $F[x]$ is the integral closure of A) we deduce that $(0:_M\mathbf{p})$ is an Artinian A-module.

It is easy to see that $\mathbf{p} = (x^3, x^4, x^5)A$ and $\mathbf{p}^2 = (x^6, x^7, x^8)A = \mathbf{a}^2$. Thus $(0:_M\mathbf{p}^2) = (0:_M\mathbf{a}^2)$ and, by Lemma 2.1 (iii) and (v), $\mathbf{p} \subseteq \mathbf{a}^* \subseteq \bar{\mathbf{a}}$. But $\bar{\mathbf{a}} \neq A$ and $\mathbf{p}$ is maximal and so we have $\mathbf{p} = \mathbf{a}^* = \bar{\mathbf{a}}$. Since $x^5 \in \mathbf{a}^*$ and $x^5 \notin \mathbf{a}$, we get $\mathbf{a} \neq \mathbf{a}^*$.

Now for $n \geq 2$, we have $\mathbf{a}^n = (x^{3n}, x^{3n+1}, x^{3n+2})A = \mathbf{p}^n = x^{3n}F[x] \cap A \subseteq \overline{x^{3n}A}$. Also $x^{3n}A \subseteq \mathbf{a}^n \subseteq \overline{x^{3n}A}$. Therefore $\overline{\mathbf{a}^n} = \overline{x^{3n}A} = \mathbf{p}^n$. Thus, by Lemma 2.1 $\mathbf{p}^n = \mathbf{a}^n \subseteq (\mathbf{a}^n)^* \subseteq \overline{\mathbf{a}^n} = \mathbf{p}^n$ and the result follows.

Finally, since $(0:_M\mathbf{p}) = (0:_M\mathbf{a}^*) = (0:_M\mathbf{a})$ and $\mathbf{a}^n = (\mathbf{a}^n)^*$ for $n \geq 2$, we get $(0:_M(\mathbf{a}^n)^*) = (0:_M\mathbf{a}^n)$ for all $n \geq 1$. Hence, by Theorem 3.2, $(T^{-1}, \mathbf{p}, \mathbf{a}T) \notin \mathrm{Att}_{\mathscr{R}}(G)$. Thus, by corollary 3.3,

$$\mathrm{Att}\,(0:_M\mathbf{a}) \subseteq \mathrm{Att}\,(0:_M\mathbf{a}^2) \subseteq \ldots .$$

Example 4.2. (See [6, §5]). Let $p_1, p_2, \ldots, p_k$ be distinct prime integers, and let $n_1, n_2, \ldots, n_k$ be positive integers. We shall let M denote the additive group modulo $p_1^{n1} p_2^{n2} \ldots p_k^{nk}$. We can regard M as a $\mathcal{Z}$-module in the usual way. Every submodule of M may be single generated, and we use $<m>$ to denote the submodule of M which is generated by the integer m of M.

By [6, §5] the secondary submodules of M are those generated by element $p_1^{\mu 1} p_2^{\mu 2} \ldots p_k^{\mu k}$, where all but one of the μ's are equal to the corresponding n, and the single exception satisfies $0 \leq \mu_i < n_i$. Since M is finite, M is an Artinian $\mathcal{Z}$-module and so each submodule of M has a secondary decomposition in M. The submodule of M generated by $p_1^{m1} \ldots p_k^{mk}$, $0 < m < n$, $1 \leq i \leq k$, has a secondary decomposition.

$$<p_1^{m1} p_2^{m2} \ldots p_k^{mk}> = \sum_{i=1}^{k} <p_1^{n1} \ldots p_{i-1}^{ni-1} p_i^{ni} p_{i+1}^{ni+1} \ldots p_k^{nk}>$$

and so $\mathrm{Att}_{\mathcal{Z}}(<p_1^{m1} p_2^{m2} \ldots p^{mk}>) = \{p_i\mathcal{Z} \mid m_i \neq n_i\}$. Also let $\mathbf{a} = <p_1 p_2 \ldots p_k>$. Then $\mathbf{a} = \mathbf{a}^* = \bar{\mathbf{a}}$ and $\mathrm{Att}\,(0:_M\mathbf{a}) = \mathrm{Att}(0:_M\mathbf{a}^2) = \ldots$.

Proof. The first part is easy. For the second part we note that if $<b>$ is a reduction of a relative to M, then it is readily seen that $<b> = \mathbf{a}$. Thus, by [10, 2.5], $\bar{\mathbf{a}} = \mathbf{a}$ and so $\mathbf{a} = \mathbf{a}^* = \bar{\mathbf{a}}$ by Lemma (2.1) (iii).

Finally, it is easy to see that $(0:_M\mathbf{a}^i) = <p_1^{n_1-i} \ldots p_k^{n_k-i}>$, (with the convention that $p_t^{n_t-i} = 1$ whenever $n_t - i < 0$.). Thus by the first part $\mathrm{Att}\,(0:_M\mathbf{a}^i) = \{p_1\mathcal{Z}, p_2\mathcal{Z}, \ldots p_k\mathcal{Z}\}$ for all $i \geq 1$.

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