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A Characterization of Square Integrable Representations Associated with CWT

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Abstract

Consider the semidirect product group $H \times_\tau K$, where H and K are two arbitrary locally compact groups and K is also abelian. We introduce the continuous wavelet transform associated to some square integrable representations $H \times_\tau K$. Moreover, we try to obtain a concrete form for admissible vectors of these integrable representations.

Keywords: Locally compact abelian (LCA) group; Continuous wavelet transform (CWT); Semidirect product; Square integrable representation

1. Introduction and Preliminaries

Wavelet analysis has found a wide range of applications in Physics, Engineering and applied mathematics in the last few years. The continuous wavelet transform on Affine and Heisenberg groups has been discussed during the last decade, but in one point of view these groups are semidirect product of two LCA groups. Many authors have considered, as a general form, the semidirect product group $H \times_\tau \mathbb{R}^n$, where H is a locally compact group.

For the reader's convenience, we provide a summary of the mathematical notations and definitions which are used in the sequel. For details we refer the reader to the general reference [12] or any standard book of harmonic analysis.

Let G be a locally compact group and $\mathcal{H}_\rho$ be a Hilbert space. A unitary representation of G is a homomorphism $\rho : G \to \mathcal{U}(\mathcal{H}_\rho)$ where $\mathcal{U}(\mathcal{H}_\rho)$ is the group of all unitary operators on $\mathcal{H}_\rho$ and ρ is continuous with respect to the strong (or weak) operator topology, namely, $x \mapsto <\rho(x)u, v>$ should be continuous from G to the complex numbers for each $u, v \in \mathcal{H}_\rho$. $\mathcal{H}_\rho$ is called the representation space of ρ, and its dimension is called the dimension of ρ. We shall consider only unitary representations. So when we say "representation" we always mean "unitary representation" unless the contrary is explicitly stated. We now introduce some standard terminology associated to unitary representations. If ρ_1 and ρ_2 are representations of G, an intertwining operator for ρ_1 and ρ_2 is a bounded linear map $T : \mathcal{H}_{\rho_1} \to \mathcal{H}_{\rho_2}$ such that $T\rho_1(x) = \rho_2(x)T$ for all $x \in G$. The set of all such operators is denoted by $\mathcal{C}(\rho_1, \rho_2)$. ρ_1 and ρ_2 are (unitary) equivalent if $\mathcal{C}(\rho_1, \rho_2)$ contains a unitary operator U, so that $\rho_2(x) = U\rho_1(x)U^{-1}$. We shall write $\mathcal{C}(\rho)$ for $\mathcal{C}(\rho, \rho)$. This is the space of bounded operators on $\mathcal{H}_\rho$ that commute with $\rho(x)$ for every

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$x \in G$; it is called the commutant of ρ. $\mathbf{C}(\rho)$ is an algebra, closed under weak limit and adjoint. In short, $\mathbf{C}(\rho)$ is a weakly closed C^*-algebra of operators on $\mathbf{H}_\rho$, that is, a von Neumann algebra. A closed subspace M of $\mathbf{H}_\rho$ is called invariant subspace for ρ if $\rho(x)M \subseteq M$ for all $x \in G$. If ρ is a unitary map of G and $u \in \mathbf{H}_\rho$, the closed linear span M_u of $\{\rho(x)u \ : \ x \in G\}$ in $\mathbf{H}_\rho$ is called the cyclic subspace generated by u. Clearly M_u is invariant under ρ. If $M_u = \mathbf{H}_\rho$, u is called a cyclic vector.

Suppose M is a non-zero closed and invariant subspace of $\mathbf{H}_\rho$, the restriction of ρ to M, $\rho_M(x) = \rho(x)\big|_M$ defines a representation of G on M, called a subrepresentation of ρ. One of the well known continuous unitary representations of a locally compact group is the left regular representation, which is defined by $L : G \to U(L^2(G))$, $y \mapsto L_y$ where $L_y f(x) = f(y^{-1}x)$ for all $x, y \in G$, and all $f \in L^2(G)$. A unitary representation ρ on a Hilbert space $\mathbf{H}_\rho$ is called irreducible if the only closed subspaces of $\mathbf{H}_\rho$ that are invariant under $\rho(x)$ for all $x \in G$ are $\{0\}$ and $\mathbf{H}_\rho$. Usually the left regular representation is not irreducible. For more details one can consult with [4].

Let G be a locally compact group, and ρ be a strongly continuous unitary irreducible representation of G on a Hilbert space $\mathbf{H}_\rho$. We wish to find a vector ψ in $\mathbf{H}_\rho$ such that

$$\int_G <\eta, \rho(x)\psi> \rho(x)\psi \, dx = \eta \quad , \ \forall \eta \in \mathbf{H}_\rho, \qquad (1)$$

where dx is the left Haar measure of G. Finding such a vector is impossible in general [13]. However, ψ satisfies (1) if and only if $\int_G |<\psi, \rho(x)\psi>|^2 \, dx < \infty$ or $\int_G |<\psi, \rho(x)\eta>|^2 \, dx < \infty$ for all $\eta \in \mathbf{H}_\rho$. Such a ψ is called an admissible vector [3]. The irreducible representation ρ is called square integrable if at least one nonzero admissible vector in $\mathbf{H}_\rho$ exists. It is easy to show that if ρ is square integrable then there exists a dense set of admissible vectors in $\mathbf{H}_\rho$. Moreover if G is unimodular then every nonzero vector is admissible and so (1) holds for all ψ in $\mathbf{H}_\rho$.

Now we fix an admissible vector ψ in $\mathbf{H}_\rho$ and define $W_\psi : \mathbf{H}_\rho \to L^2(G)$ as follows:

$$(W_\psi \eta)(g) = C_\psi^{\frac{-1}{2}} <\eta, \rho(g)\psi> \quad , \eta \in \mathbf{H}_\rho \ , \ g \in G$$

where $C_\psi = \frac{1}{\|\psi\|^2} \int_G |<\psi, \rho(g)\psi>|^2 \, dg$. W_ψ is called the continuous wavelet transform associated to ρ and has the following properties [3,8]:

- W_ψ is a linear isometry onto a closed subspace, denoted by $\mathbf{H}_\psi$, of $L^2(G)$.
- W_ψ intertwines ρ and the left regular representation L, (*i.e.* $W_\psi \rho(g) = L_g W_\psi \quad \forall g \in G$)
- The adjoint of W_ψ and W_ψ^{-1} coincide on $\mathbf{H}_\psi$.
- The Inversion formula holds *i.e.*

$$\int_G (W_\psi \eta)(g) \, \rho(g)\psi \, dg = C_\psi \eta \qquad \textit{for all } \eta. \quad (2)$$

Define $(f * g)(x) = \int_G f(y) g(y^{-1}x) dx$ whenever it makes sense. For a locally compact abelian group G, the set of all continuous characters (group homomorphisms from G to $\mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}$) of G is called the dual group of G. When it is equipped with the topology of uniform convergence on compact subsets of G, it is a locally compact abelian group, denoted by $\hat{G}$. For example:

i) Let $\mathbb{R}$ be the set of all real numbers with the usual topology. $(\mathbb{R}, +)$ is a locally compact unimodular group whose Haar measure is the Lebesgue measure. $\hat{\mathbb{R}} \cong \mathbb{R}$, with the pairing $<x, \gamma> = e^{2\pi i \gamma x}$, so the Lebesgue measure is self-dual.

ii) The groups $\mathbb{T}$ and $\mathbb{Z}$ are dual to each other; the natural dual measures on them are the normalized Lebesgue measure $\frac{d\theta}{2\pi}$ and counting measure on $\mathbb{Z}$, respectively.

For $f \in L^1(G) \cap L^2(G)$, the function $\hat{f}$ is defined on $\hat{G}$ by $\hat{f}(\gamma) = \int_G f(x) \overline{\gamma(x)} dx$ and is called the Fourier transform of f. We conclude this section with two well-known theorems.

Theorem 1.1. ***(Plancherel's theorem)*** *The Fourier transform on $L^1(G) \cap L^2(G)$ extends uniquely to a unitary isometry from $L^2(G)$ to $L^2(\hat{G})$. More precisely, there exists a measure $\mu_{\hat{G}}$ on $\hat{G}$ (the dual measure) such that for every $f \in L^2(G)$, one has $\hat{f} \in L^2(\hat{G})$ and $\|f\|_2 = \|\hat{f}\|_2$. Also the inverse Fourier transformation is a linear isometry of $L^2(\hat{G})$ onto*

$L^2(G)$. These two transformations are inverses of each other.

Corollary 1.2. ***(Fourier Inversion Formula)*** *If $f \in L^1(G)$ and $\hat{f} \in L^1(\hat{G})$ then $f(x) = (\hat{f})\hat{}(x^{-1})$ for a.e. x; that is,*

$$f(x) = \int_{\hat{G}} \hat{f}(\gamma)\gamma(x)d\mu_{\hat{G}}(\gamma),$$

for a.e. $x \in G$. If f is continuous, these relations hold for every $x \in G$.

For example when G is the Affine group (which is the semidirect product of the two LCA groups, $(\mathbb{R}\setminus\{0\},.)$ and $(\mathbb{R},+)$), the group operations on G are

$$(a,b)(c,d) = (ac, b+ad) \;\;,\;\; (a,b)^{-1} = (\frac{1}{a}, \frac{-b}{a}).$$

Now we define the quasi regular representation ρ of G on the Hilbert space $L^2(\mathbb{R})$ by $(\rho(a,b)f)(x) = |a|^{\frac{-1}{2}} f(\frac{x-b}{a})$. Then ρ is an irreducible representation of G. Moreover a vector $\psi \in L^2(\mathbb{R})$ is an admissible vector for ρ if $C_\psi^2 = \int_{\mathbb{R}} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty$, [13]. Hence for a wavelet vector ψ we can define the CWT W_ψ by $W_\psi f(a,b) = < f, \rho(a,b)\psi >$ where $f \in L^2(\mathbb{R})$, $a \in \mathbb{R}\setminus\{0\}$, $b \in \mathbb{R}$ and rewrite the Inversion formula (2) as follows (for more details see [13]):

$$\int_{\mathbb{R}\setminus\{0\}}\int_{\mathbb{R}} < f, \rho(a,b)\psi > \rho(a,b)\psi \frac{dadb}{a^2} = C_\psi f.$$

In this article we study these properties for the groups which are semidirect product of two locally compact groups. One should note that the above representation ρ is not irreducible in general (for example in the Affine group by replacing $\mathbb{R}\setminus\{0\}$ with positive real numbers $\mathbb{R}^+$). We can overcome this problem by taking an irreducible subrepresentation ρ instead. Furthermore we investigate when this subrepresentation is square integrable and how to obtain wavelet vectors. In most cases the general form of the Affine group is a semidirect product where is a closed subgroup (not necessarily abelian) of H. Fuhr and S.T. Ali provide a comprehensive study of CWT on these groups [5,1].

The affine group and Heisenberg group are two simple and essential examples of the semidirect product groups, (see [9,10]). In Sections 3, we overview these cases. Afterwards we try to extend the notion of continuous wavelet transform to the semidirect product of two locally compact topological groups.

2. Wavelet Transform on Semidirect Product Groups

Throughout this section we assume that H and K are two locally compact topological groups and K is abelian. Now if there exists a homomorphism $h \mapsto \tau_h$ from H to $Aut(K)$ such that $(h,x) \mapsto \tau_h(x)$ is continuous from $H \times K$ into K, then $H \times K$ under operations:

$$(h,x).(h',x') = (hh', x.\tau_h(x'))$$

$$(h,x)^{-1} = (h^{-1}, \tau_{h^{-1}}(x^{-1})),$$

is a locally compact topological group with the product topology. This group, denoted by $H \times_\tau K$, is called the semidirect product of H and K, respectively. Take $G = H \times_\tau K$. If $d\mu(h) = dh$ and $d\nu(x) = dx$ are the left Haar measures on H and K, respectively and Δ_H is the modular function of H, then $\delta(h)dhdx$ is the left Haar measure of G where $\delta : H \to (0,+\infty)$ is a monomorphism such that:

$$\int_K (f \circ \tau_h)(x)dx = \delta(h)\int_K f(x)dx \;\;,$$

$$\forall h \in H, \forall f \in C_{00}(K).$$

Also $\Delta(h,x) = \delta(h)\Delta_H(h)$ is the modular function of G, (for more details see [12]).

Lemma 2.1. With the notations as above, the action of G on K which is given by $T_{h,x}(y) = \tau_h(y).x$ for $(h,x) \in G$, induces a unitary representation (is called the quasi regular representation) ρ of G on $L^2(K)$, that is defined by

$$\rho(h,x)f(y) = \delta(h)^{\frac{1}{2}} f(T_{h,x}^{-1}(y)) \;\; \forall f \in L^2(K).$$

Proof. First note that $T^{-1}{}_{h,x}(y) = \tau_{h^{-1}}(yx^{-1})$. So we have $\rho(h,x)f(y) = \delta(h)^{\frac{1}{2}} f(\tau_{h^{-1}}(yx^{-1}))$ for all $f \in L^2(K)$. Thus

$$\begin{aligned}
&\rho(h_1,x_1)\rho(h_2,x_2)f(y)\\
&=\rho(h_1,x_1)(\delta(h_2)^{\frac{1}{2}}f(\tau_{h_2^{-1}}(yx_2^{-1})))\\
&=\delta(h_1)^{\frac{1}{2}}.\delta(h_2)^{\frac{1}{2}}f(\tau_{h_2^{-1}}(\tau_{h_1^{-1}}(yx_1^{-1})x_2^{-1}))\\
&=\delta(h_1h_2)^{\frac{1}{2}}f(\tau_{(h_1h_2)^{-1}}(x_1^{-1}y).\tau_{h_2^{-1}}(x_2^{-1}))\\
&=\delta(h_1h_2)^{\frac{1}{2}}f(\tau_{(h_1h_2)^{-1}}(x_1^{-1}.\tau_{h_1}(x_2^{-1})y))\\
&=\rho((h_1,x_1)(h_2,x_2))f(y).
\end{aligned}$$

Moreover

$$\begin{aligned}
\|\rho(h,x)f\|_2^2 &= \int_K|\rho(h,x)f(y)|^2\,dy\\
&=\int_K\delta(h)\left|f(\tau_{h^{-1}}(yx^{-1}))\right|^2dy\\
&=\int_K\left|f(yx^{-1})\right|^2dy=\int_K|f(y)|^2\,dy\\
&=\|f\|_2^2.
\end{aligned}$$

□

Definition 2.2. A nonzero vector ψ in $L^2(K)$ is called a wavelet (or equivalently an admissible vector for ρ) if $<f,\rho(.,.)\psi>\in L^2(G)$ for all $f\in L^2(K)$.

When K is a LCA group, the identity $\hat{\rho}(h,x)\hat{f}=(\rho(h,x)f)\hat{}$ defines the Fourier transform of the quasi regular representation ρ. Moreover for any $f\in L^2(K)$ we have:

$$\begin{aligned}
\hat{\rho}(h,x)\hat{f}(\gamma)&=(\rho(h,x)f)\hat{}(\gamma)\\
&=\int_K\rho(h,x)f(y)\overline{\gamma(y)}dy\\
&=\delta(h)^{\frac{1}{2}}\int_K f(\tau_{h^{-1}}(y)).\overline{\gamma(yx)}dy\\
&=\delta(h)^{\frac{1}{2}}\overline{\gamma(x)}\int_K f(\tau_{h^{-1}}(y)).\overline{\gamma(y)}dy\\
&=\delta(h)^{\frac{1}{2}}\overline{\gamma(x)}(f\circ\tau_{h^{-1}})\hat{}(\gamma).
\end{aligned}$$

Now for wavelet vector ψ and $f\in L^2(K)$, we define the continuous wavelet transform of f by

$$W_\psi f(h,x)=<f,\rho(h,x)\psi>=\delta(h)^{\frac{1}{2}}\int_K f(y)\overline{\psi(\tau_{h^{-1}}(yx^{-1}))}\,dy.$$

It is easy to see that $(h,x)\mapsto W_\psi f(h,x)$ is a continuous mapping on G. Also W_ψ intertwines ρ and the left regular representation L of G, *i.e.* $W_\psi\rho(h,x)=L(h,x)W_\psi$ for all $(h,x)\in G$. Now we are ready to obtain a concrete form for admissibility of ψ in the following theorem;

Theorem 2.3. Let ρ be the quasi regular representation of $G=H\times_\tau K$ and $\psi,f\in L^2(K)$.

i) *If ψ is a wavelet vector and $d\gamma$ is the Haar measure of $\hat{K}$, then*

$$W_\psi f(h,x)=\delta(h)^{\frac{1}{2}}\int_{\hat{K}}\hat{f}(\gamma).\overline{(\psi\circ\tau_{h^{-1}})\hat{}(\gamma)}\gamma(x)d\gamma.$$

ii) *ψ is a wavelet vector if $C_\psi^2:=\int_H|\hat{\psi}(\gamma\circ\tau_h)|^2dh<\infty$. Moreover, in this case $\|W_\psi f\|_2=C_\psi\|f\|_2$.*

Proof. (i) Applying the Plancherel theorem for $f\in L^2(K)$ we have:

$$\begin{aligned}
W_\psi f(h,x)&=<f,\rho(h,x)\psi\\
&>=\delta(h)^{\frac{1}{2}}\int_{\hat{K}}\hat{f}(\gamma).\overline{(\psi\circ\tau_{h^{-1}})\hat{}(\gamma)}\gamma(x)d\gamma.
\end{aligned}$$

(ii) If we define $F=f*(\psi\circ\tau_{h^{-1}})^{\bullet}$ where $g^{\bullet}(x)=\overline{g(x^{-1})}$ and $(g^{\bullet})\hat{}=\overline{\hat{g}}$ for any $g\in L^1(K)$.

Then $\hat{F}=\hat{f}.\overline{(\psi\circ\tau_{h^{-1}})\hat{}}\in L^1(\hat{K})$. (see Proposition 4.13 of [4]). Also by Theorem 4.32 of [4] we obtain $F(x)=\int_{\hat{K}}\hat{F}(\gamma)\gamma(x)d\gamma$. Hence for any $f\in L^2(K)$ we have:

$$\begin{aligned}
&\int_K|W_\psi f(h,x)|^2dx\\
&=\int_K W_\psi f(h,x)\overline{W_\psi f(h,x)}dx\\
&=\delta(h)\int_K\int_{\hat{K}}\int_{\hat{K}}\hat{f}(\gamma).\overline{(\psi\circ\tau_{h^{-1}})\hat{}(\gamma)}\hat{f}(\omega).\\
&\qquad\qquad(\psi\circ\tau_{h^{-1}})\hat{}(\omega)\gamma(x)\overline{\omega(x)}d\gamma d\omega dx\\
&=\delta(h)\int_K\int_{\hat{K}}\int_{\hat{K}}\hat{F}(\gamma)\gamma(x)\overline{\hat{F}(\omega)}\overline{\omega(x)}d\gamma d\omega dx
\end{aligned}$$

$$= \delta(h)\int_K |F(x)|^2\,dx$$

$$= \delta(h)\int_{\hat{K}} |\hat{F}(\gamma)|^2\,d\gamma$$

$$= \delta(h)\int_{\hat{K}} |\hat{f}(\gamma)|^2\,|(\psi\circ\tau_{h^{-1}})\hat{}\,|^2(\gamma)\,d\gamma.$$

A straightforward calculation gives:

$$(f\circ\tau_{h^{-1}})\hat{}(\gamma) = \delta(h)^{-1}\hat{f}(\gamma\circ\tau_h),\ \forall f \in L^2(K). \quad (3)$$

So that;

$$\|W_\psi f\|_2^2 = \int_{H\times_\tau K} |W_\psi f(h,x)|^2\,\delta(h)\,dx\,dh$$

$$= \int_H \delta(h)\int_K |W_\psi f(h,x)|^2\,dx\,dh$$

$$= \int_H \delta(h)^2\int_{\hat{K}} |\hat{f}(\gamma)|^2 . |(\psi\circ\tau_{h^{-1}})\hat{}(\gamma)|^2\,d\gamma\,dh$$

$$= \int_{\hat{K}} |\hat{f}(\gamma)|^2\int_H |(\psi\circ\tau_{h^{-1}})\hat{}(\gamma)|^2\,dh\,d\gamma$$

$$= C_\psi^2\,\|f\|_2^2.$$

□

To obtain an irreducible subrepresentation of ρ, we define an action from H on $\hat{K}$ by $(h,\gamma)\mapsto\gamma\circ\tau_h$. This action plays an essential role in our work. In what follow, we consider only this action on $\hat{K}$, and the Haar measure of $\hat{K}$ is dual of the Haar measure of K.

Notation 2.4. For any measurable subset A of $\hat{K}$ with positive measure, put

$$L_A^2(K) = \{f \in L^2(K)\ ;\ Supp(\hat{f}) \subseteq A\}.$$

It is easy to show that $L_A^2(K)$ is a closed subspace of $L^2(K)$ and is invariant under translation. Now we are going to show the converse.

Lemma 2.5. Let G be a LCA group and N a closed subspace of $L^2(G)$. For a measurable function h let M_h be the multiplication operator by h. If P is an orthogonal projection onto N such that $PM_h = M_hP$ for all bounded measurable function h, then $P = M_{\chi_A}$ for some measurable subset A of G.

Proof. Choose a bounded function $h\in L^2(G)$ such that $h(x)>0$ for all x, and define $\phi(x) = \frac{(Ph)(x)}{h(x)}$. Then $PM_{\chi_E}h = M_\phi M_{\chi_E}h$ for all measurable sets E having finite measure. Thus the identity $P = M_\phi$ follows from the fact that linear combinations of $M_{\chi_E}h$ is dense in $L^2(G)$. On the other hand P is an orthogonal projection and hence $P = M_{\chi_A}$ for some measurable set A of G with positive measure. □

Theorem 2.6. *i) Translation invariant closed subspaces of* $L^2(K)$ *are precisely* $L_A^2(K)$, *for measurable subsets A of* $\hat{K}$.

ii) If A *is a measurable invariant set in* $\hat{K}$, *Then* $L_A^2(K)$ *is* $\rho-$*invariant. (so in this case the restriction* ρ *on* $L_A^2(K)$, *denoted by* ρ_A, *is a subrepresentation of* ρ*).*

Proof. (i) Let $N = \hat{M}$, where M is a translation invariant closed subspace of $L^2(K)$, then N is invariant under the operators $\hat{f}\mapsto(L_x f)\hat{}$, for all $x\in K$. Moreover if $h\in L^1(K)$ then

$$\hat{h}(\gamma).\hat{f}(\gamma) = \int_K h(x)\overline{\gamma(x)}\hat{f}(\gamma)\,dx$$

$$= \int_K h(x^{-1})(L_x f)\hat{}(\gamma)\,dx.$$

Now by Theorem 3.12 of [4] every multiplication operator $M_{\hat{h}} : L^2(\hat{K})\to L^2(\hat{K})$ where $h\in L^1(K)$ can be weakly approximated by a finite linear combination of the operators $\hat{f}\mapsto(L_x f)\hat{}$, and so N is invariant under $M_{\hat{h}}$ for all $h\in L^1(K)$. By continuity of Fourier transform, N is invariant under M_h for all $h\in C_0(\hat{K})$. Hence by the approximation theory, N is invariant under M_h for all bounded and measurable functions h. So by lemma 2.5, with $G=\hat{K}$ we have $N = \{\chi_A . f\ ;\ f\in L^2(K)\}$ for some measurable set A *of* $\hat{K}$ with positive measure. Thus $M = L_A^2(K)$. (ii) follows immediately from (3). □

Definition 2.7. An invariant measurable subset A of $\hat{K}$ *with positive measure* is called ergodic if every invariant subset of A is null or conull.

Recall that ergodicity can be defined for actions on measure spaces and there is a well-known fact related to existence of irreducible subrepresentations. In fact ergodicity of A and Theorem 2.7 guarantee that there isn't any nontrivial invariant closed subspace of $L_A^2(K)$ under representation ρ_A and so it is irreducible. Hence we have proved the following result.

Corollary 2.8. A nonzero closed subspace M of $L_A^2(K)$ is invariant under the representation ρ if and only if $M = L_A^2(K)$ for some measurable invariant subset A of $\hat{K}$ with positive measure. Moreover the subrepresentation ρ_A is irreducible if A is ergodic.

Now we have an irreducible representation and the next theorem shows when an admissible vector exists. This vector will be a wavelet and so we can define continuous wavelet transform.

By the orbit of γ we mean the ergodic set $O_\gamma = \{\gamma \circ \tau_h \ ; \ h \in H\}$. An ergodic set cannot contain two disjoint orbits with positive measure. So if A is ergodic such that O_γ has positive measure for some $\gamma \in A$, then $A = O_\gamma$ a.e. Now we can summarize some results in the following theorem:

Theorem 2.9. *Let* $G = H \times_\tau K$ *be the semidirect product of* H *and* K, *also let* $A \subseteq \hat{K}$ *be an ergodic set with positive measure such that* $A = O_\gamma$ *a.e. for some* γ. *Then:*

i) The representation ρ_A *is square integrable if* H^γ *(the stabilizer of* γ*) is compact.*

ii) $\psi \in L_A^2(K)$ *is admissible if* $h \mapsto \hat{\psi}(\omega \circ \tau_h)$ *is in* $L^2(H)$ *for almost all* $\omega \in A$.

Proof. (i) By Corollary 2.8 the subrepresentation ρ_A is irreducible. Moreover the Fourier transform from $L^2(K)$ onto $L^2(\hat{K})$ intertwines ρ_A with $Ind_{G_0}^G \gamma I$ where I is the identity operator and $\gamma I(h,x) = \gamma(x) I$ is a representation on subgroup $G_0 = H^\gamma \times_\tau K$. Now since $\mu(A) > 0$, by Theorem 2 of [2] the representation ρ_A is square integrable if and only if I is square integrable or equivalently the subgroup H^γ is compact.

(ii) For any $\omega \in A$ we define $F(\omega) := \int_H |\hat{\psi}(\omega \circ \tau_a)|^2 \, da$, then

$$F(\omega \circ \tau_h) = \int_H |\hat{\psi}(\omega \circ \tau_h \circ \tau_a)|^2 \, da$$

$$= \int_H |\hat{\psi}(\omega \circ \tau_{ha})|^2 \, da = F(\omega).$$

Hence F is constant on A. So there is a constant $C_\psi^2 \le \infty$ such that

$$\int_H |\hat{\psi}(\omega \circ \tau_a)|^2 \, da = C_\psi^2 \quad a.e. \ on \ A.$$

Now Theorem 2.3 completes the proof. □

Remark 2.10. *Theorem 2.9(ii) gives a necessary and sufficient condition for the existence of admissible (wavelet) vectors, and if* $\psi \in L_A^2(K)$ *is a wavelet then we can rewrite the reproducing formula (1):*

$$\int_G < f, \rho(h,x)\psi > \rho(h,x)\psi \, \delta(h) \, dhdx = C_\psi f$$

(interpreted in the weak sense) for all $f \in L_A^2(K)$. *The direct construction of admissible vectors will be difficult, in fact these are not compactly supported in general. Although for a square integrable representation on unimodular groups every vector is admissible (Example3.4), see also Theorem 7.29 of [7].*

3. Applications and Examples

We present a list of examples to illustrate when the quasi regular representation has a square integrable subrepresentation and how our result can be applied.

Example 3.1. ***Affine groups***

Assume that $G = (\mathbb{R} \setminus \{0\}) \times_\tau \mathbb{R}$ is a class of Affine groups such that homomorphism $h \mapsto \tau_h$ from $\mathbb{R} \setminus \{0\}$ onto $Aut(\mathrm{R})$ is defined by $\tau_h(x) = hx$. Then $\delta(h) = h^{-1}$ and the quasi regular representation ρ is given by $\rho(h,x) f(y) = |h|^{\frac{-1}{2}} f(\frac{y-x}{h})$, for all $f \in L^2(\mathbb{R})$. Moreover $\psi \in L^2(\mathbb{R})$ is a wavelet vector if $\int_G |< f, \rho(h,x)\psi >|^2 \frac{dhdx}{h^2} < \infty$ or equivalently $C_\psi^2 = \int_{\mathbb{R} \setminus \{0\}} \frac{|\hat{\psi}(h)|^2}{|h|} dh < \infty$.

In this case $\hat{\mathbb{R}}$ is an ergodic sets in itself, hence Corollary 2.8 implies that the representation ρ is irreducible. Also for any nonzero $y \in \hat{\mathbb{R}}$ the stabilizer of y is compact and the orbit of y has positive measure, hence ρ is square integrable. Furthermore if we replace $\mathbb{R} \setminus \{0\}$ with $\mathbb{R}^+$ then ρ is not irreducible, since $L_{A_1}^2(\mathbb{R})$ and $L_{A_2}^2(\mathbb{R})$ are invariant closed subspaces of $L^2(\mathbb{R})$ where $A_1 = [0, +\infty)$ and $A_2 = (-\infty, 0]$. But in this case it can be similarly shown that the subrepresentations ρ_{A_1} and ρ_{A_2} are square integrable [11].

Example 3.2. ***Motion group***

For any $h \in SO(n, \mathbb{R})$ assume that τ_h, an automorphism on $\mathbb{R}^n$, is defined by $\tau_h(x) = hx$, then the semidirect product $G = SO(n, \mathbb{R}) \times_\tau \mathbb{R}^n$ is isomorphic to the group of motion of the n-dimensional Euclidean space. It is easy to see that for every $y \in (\mathbb{R}^n \hat{)}$ the orbit O_y is a null set and there exists no ergodic subset in $(\mathbb{R}^n \hat{)}$, so the quasi regular representation ρ doesn't have any such irreducible subrepresentation. However, in this case invariant subsets of $(\mathbb{R}^n \hat{)}$ exist, and so ρ have reducible subrepresentations. Laugesen in [14] showed that the quasi regular representation on $H \times_\tau \mathbb{R}^n$ where H is a compact group, is not square integrable.

Example 3.3.

Let $H = \mathbb{R} \setminus \{0\} \times_\tau \mathbb{R}$ be a type of Affine group, one can define the semidirect product $G = H \times_\tau \mathbb{R}^2$ where automorphism $\tau_{(a,b)}$ on $\mathbb{R}^2$ is defined by $\tau_{(a,b)}(x, y) = (ax, bx + y)$. Moreover for any $\gamma = (\gamma_1, \gamma_2) \in (\mathbb{R}^2 \hat{)}$ we have:

$$\begin{aligned} H^\gamma &= \{(a,b) \in H \ ; \ (\gamma_1, \gamma_2) \circ \tau_{(a,b)} = (\gamma_1, \gamma_2)\} \\ &= \{(a,b) \in H \ ; \ (a\gamma_1, b\gamma_1 + \gamma_2) = (\gamma_1, \gamma_2)\} \\ &= \begin{cases} H & \text{for } \gamma_1 = 0 \\ \{(1,0)\} & \text{for } \gamma_1 \neq 0. \end{cases} \end{aligned}$$

Also

$$\begin{aligned} O_\gamma &= \left\{(\gamma_1, \gamma_2) \circ \tau_{(a,b)} \ ; \ (a,b) \in H\right\} \\ &= \begin{cases} \{\gamma\} & \text{for } \gamma_1 = 0 \\ (\mathbb{R} \setminus \{0\} \times \mathbb{R}) & \text{for } \gamma_1 \neq 0. \end{cases} \end{aligned}$$

Thus H^γ is compact and O_γ has positive measure a.e. $\gamma \in (\mathbb{R}^2 \hat{)}$, and so for almost all $\gamma \in (\mathbb{R}^2 \hat{)}$ the subrepresentations associated to γ are square integrable.

Example 3.4. ***Heisenberg group***

Let $H(\mathbb{R}^n) = \mathbb{R}^n \times_\tau (\mathbb{R}^n \hat{)} \times \mathbb{T}$ be the Heisenberg group on $\mathbb{R}^n$, when the continuous homomorphism $x \mapsto \tau_x$ from $\mathbb{R}^n$ into $Aut((\mathbb{R}^n \hat{)} \times \mathbb{T})$ is defined by $\tau_x(y, \xi) = (y, \xi e^{2\pi i x . y})$. This group is unimodular (see [11]) and a type of the quasi regular representation of $H(\mathbb{R}^n)$ on $L^2(\mathbb{R}^n)$ is given by

$$\pi(a, b, t) f(x) = t e^{2\pi i b(x-a)} f(x - a).$$

In [11] it is directly shown that π is square integrable and every $g \in L^2(\mathbb{R}^n)$ is admissible. To see how our machinery works, first we recall that the dual group of $(\mathbb{R}^n \hat{)} \times \mathbb{T}$ is isomorphic with $\mathbb{R}^n \times \mathrm{Z}$, hence for any $(y, m) \in \mathbb{R}^n \times \mathbb{Z} \setminus \{0\}$ we have:

$$\begin{aligned} \mathbb{R}^{n(y,m)} &= \{x \in \mathbb{R}^n \ ; \ (y, m) \circ \tau_x \\ &= (y, m)\} = \{x \in \mathbb{R}^n \ ; \ y + mx = y\} \\ &= \{(0, 0, \ldots, 0)\} \end{aligned}$$

$$\begin{aligned} O_{(y,m)} &= \{(y, m) \circ \tau_x \ ; \ x \in \mathbb{R}^n\} \\ &= \{(y + mx, m) \ ; \ x \in \mathbb{R}^n\} \\ &= \mathbb{R}^n \times \{m\} \end{aligned}$$

So the stabilizers are compact and the orbits have positive measure a.e. on $\mathbb{R}^n \times \mathbb{Z}$. Now fix m and put $A = \mathbb{R}^n \times \{m\}$ then by Theorem 2.9 ρ_A is square integrable and for every $\psi \in L_A^2((\mathbb{R}^n \hat{)} \times \mathbb{T})$ we have:

$$\begin{aligned} C_\psi &= \int_{\mathrm{R}^n} \left|\hat{\psi}((y, m) \circ \tau_x)\right|^2 dx \\ &= \int_{\mathrm{R}^n} \left|\hat{\psi}(y + mx, m)\right|^2 dx = \tfrac{1}{m} \|\psi\|_2^2 < \infty \end{aligned}$$

i.e. ψ is an admissible vector by Theorem 2.9, therefore we get the same results if we consider every element of $L^2((\mathbb{R}^n \hat{)})$ as the element of $L^2((\mathbb{R}^n \hat{)} \times \mathbb{T})$ that is constant on $\mathbb{T}$.

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