

Control Problem and its Application in Management and Economics

By:

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Abstract

The control problem and Dynamic programming is a powerful tool in economics and management. We review the dynamic programming problem from its beginning up to its present stages. A problem which was involved in physics and mathematics in 17th century led to a branch of mathematics called calculus of variation which was used in economic, and management at the end of the first quarter of the 20th century. This branch of Mathematics stated its actual development under the name control problem from the second half of 20th century. Its solution was made possible through the dynamic programming method, and maximum principle. Experts in economics and management started using these methods. Then In various works from the 1970s. The stochastic optimum control was used. In this paper we will consider from the first article in maximizing the profit of a firm up to its recent applications in economics and management.

Key words: Static programming, Dynamic programming, Calculus variation, Control problem, Maximum principle and stochastic optimal control.

Introduction

In the 17th century a physics problem was brought up; if an object with Weight m slides, only under the force of gravity, from a fixed point A to a fixed point B , what would be the path of movement which results in least time-consuming path? Bernoulli Brothers solved this problem and built the base of a discussion that created the calculus of variations branch in mathematics later on. Later Euler solved the problem in a more general way and Legendre found its sufficient condition. Mathematicians improved this branch over the years. Its first use in economics dates back to the late 1920s and early 1930s by Evince^[1] Ramsey^[2] and Hotelling^[3]. During the later years this issue was mentioned in articles every now and then. Modern topics started in the 1950s by

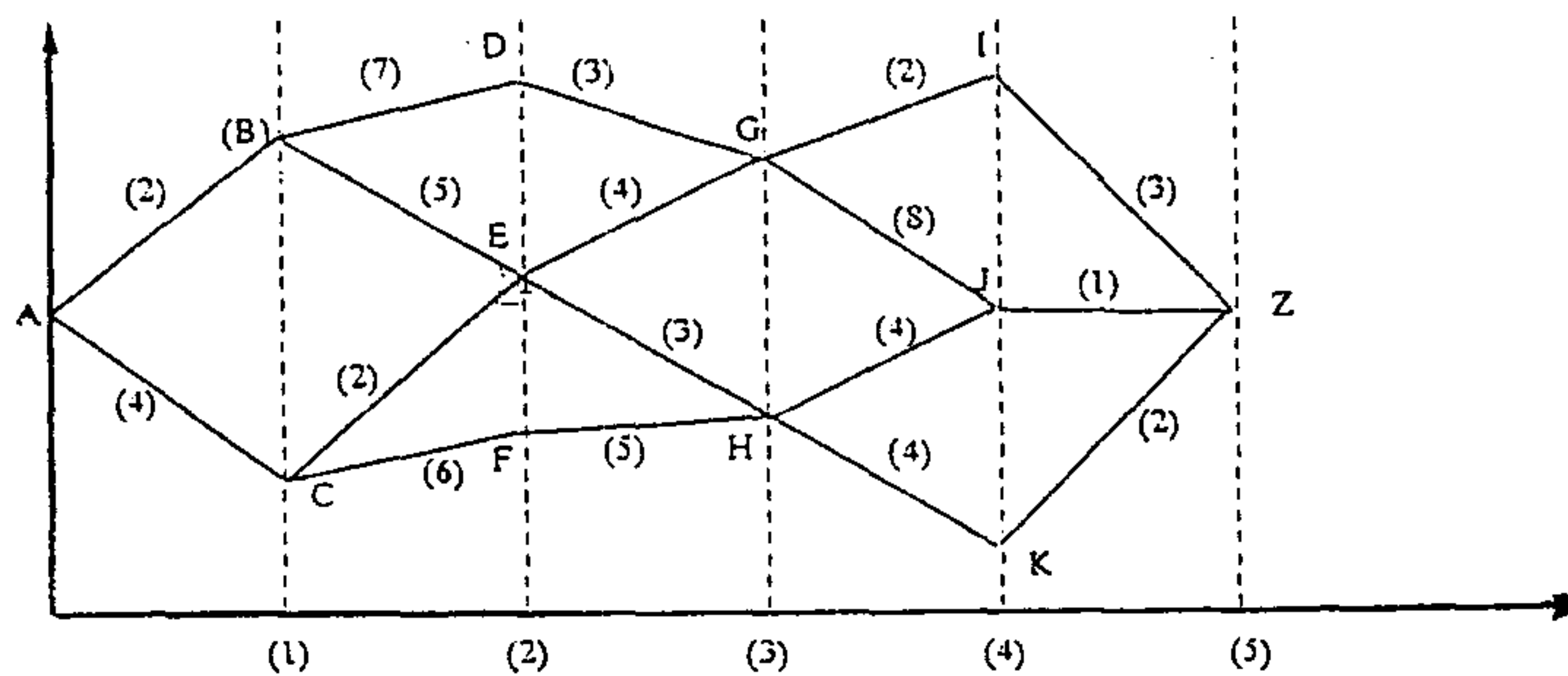
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mathematicians. Dynamic Programming (D.P.) was developed by Richard Bellman^[4] in the 1950s. Optimal Control (O.C.) was developed in the late 1960s by Pontryagin^[5] and his colleagues in Russia, and in 1962 it was translated into English. This method is actually the extension of calculus of variation. Control Problem plays a special role in guiding missiles and spaceships. During these years, the economists who were interested in the models of Optimal Growth Economy which (Ramsey), and experts of management who favored Dynamic Programming in discrete form used these methods. Today these methods have a wide application in economics and management and have attracted the attention of economics and management experts in Iran. In this article we will begin our discussion with the review of the related literature on the subject and show cases of its application in economic and management in different fields from the early stages up to the present.

1- Multistage Decision Making and its Limit

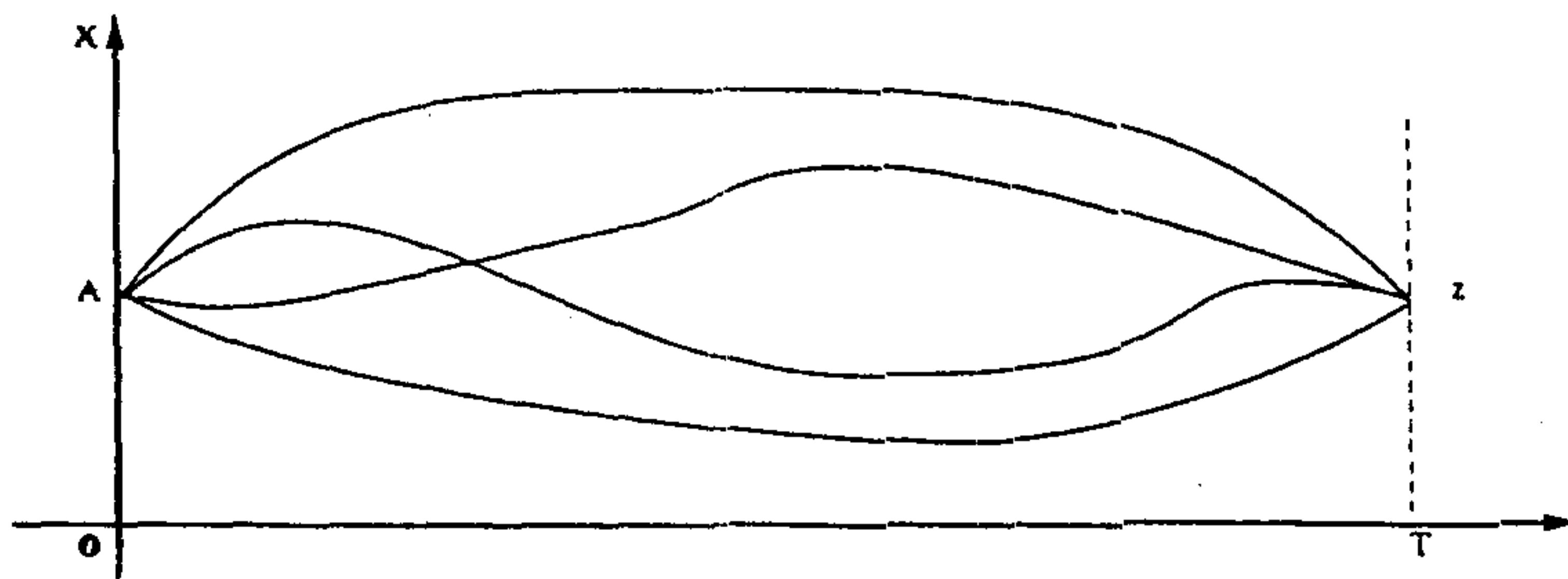
Dynamic Programming can be regarded as a continuous aspect of Multistage Decision Making which, we also face with this subject in management. Let us that a firm engages in transforming a certain substance from an initial state *A* (raw material state) into terminal state *Z* (finished product state), through a five-stage production process. In every stage, the firm faces the problem of choosing among several possible alternative sub processes, each entailing a specific cost. In this case what is the best path? Look at figure 1 below in which the horizontal axis represents “stage” and vertical axis stands for “state”.

Figure 1:



The cost of each path is shown on it. Now the production manager wants to find the least-cost path from point A to point Z . By using DP (in a separate mode) we can determine the optimal path. The problem is very easy and the optimal path is ACEHJMZ, the least-cost of which is $V(P^\circ) = 16$. Now let's say the number of stages ($N=5$) approaches the infinity; in fact, picture 1 changes to figure 2 below, discrete paths convert to continuous paths and the stages stand for the time variable.

figure 2:



It is obvious that we can move from A to Z in an infinite number of ways. But what is the optimal path? In general, this problem is refined to as Control Problem which is the concern of what follows.

2- Mathematical Statement of C.P.

In C.P. we are faced with time, the state variables, and the control variables the equation of motion, the determination of terminal time, and the objective function. With the time variable (t) which continuously changes in the Interval $[t_0, t_1]$ where, t_0 is initial and t_1 is terminal time, such that:

$$t_0 \leq t \leq t_1$$

For a given, t belonging to $[t_0, t_1]$ the vector X will indicate the state of the system and has n component and is called state variable, is equal to:

$$X = (x_1(t), x_2(t), \dots, x_n(t))' \in R^n \tag{1}$$

The components of the state variable are continuous functions of time, so the state path would be $\{X(t)\} \in E^n$ and the state variable belongs to the Fissible space of A . Geometrically, the state the state path in $E^{(n)}$, extends from

$X(t_0) = X_0$. To $X(t_1) = X_1$, which X_0 and X_1 are called the initial state (and often is known), and terminal state respectively. The third variable $U(t)$ is the control variable which means it is a vector in the m dimension space equal to:

$$U(t) = (U_1(t), U_2(t), \dots, U_m(t)) \in R^m \quad (2)$$

$U(t)$ is a vector and need to be Piece wise continuous function of time, the state variable $X(t)$ is influenced by the control variable $U(t)$. The control path would be as follows:

$$\{U(t)\} \in E^r$$

Control variables are subset from a given set BCE^r . State variables are related to the control variables through the Motion Equation. In general, the rate of change of the state variable respect to the time is a function of the control, state and time variables differential equation, which can he expressed in the following:

$$\dot{X}(t) = f(X(t), U(t), t) \quad (3)$$

The objective functional is a mapping from control paths, state paths, time, terminal time and terminal state to a point on the real exes, which is to be maximized, such that:

$$J = \int_{t_0}^{t_1} I(X(t), U(t), t) dt + F(x_1, t_1) \quad (4)$$

In the above Integral I is the Intermediate function and F is the final function. Generally, C.P. can be expressed as the following:

$$\max J = \int_{t_0}^{t_1} I(X, U, t) dt + F(x_1, t_1) \quad (5)$$

Subject to:

$$\dot{X}(t) = f(X, U, t)$$

$$x_0 \text{ and } t_0 \text{ are given, } (x(t), t) \in T \text{ at } t = t_1 \quad U(t) \in B.$$

In fact C.P. is finding the optimal control $\{U^*(t)\}$ which is determined through the optimal path $\{X^*(t)\}$ such that the functional objective would be maximized.

This C.P. is called Bolza Control problem. If $\dot{X}(t) = U(t)$ and $F = 0$, then C.P. is known as Lagrange problem. If $I = 0$, then C.P. is called Mayra Control problem.

3- Comparison between static programming and Dynamic Programming

In static programming, we want to choose the decision variable $X \in \mathbb{R}^n$ from the feasible space of A such that the objective function $Z = f(x)$ to be maximized thus:

$$\begin{aligned} \text{Max } Z &= f(x) \\ \text{s.t. } X &\in A \end{aligned} \tag{6}$$

The solution is a point $X^* \in \mathbb{R}^n$.

This optimization is independent of time; however, in dynamic programming all variable S are functions of time and the optimal path $\{X^*(t)\}$ is obtained by control problem (5). This path has two dimensions in picture (2) and in general we want to choose the optimal paths $X(t)$ or the control variable $U(t)$ in such a way that Functional (5) would be maximized or minimized.

If C.P. is in the form of Lagrange, the problem can be solved through Calculus of variations^[6]. But if the problem is a Bolza or Mayra one, it would be possible to solve it based on Dynamic Programming or Pontryagin's Maximum Principle.

4- Solving the C.P. by the Calculus of Variations

If C.P. 5 is as below, then it is called the Lagrange control problem

$$\text{Max } J = \int_{t_0}^{t_1} I(X, \dot{X}, t) dt \tag{7}$$

In this case the Motion Equation is as $\dot{X} = U$ which substitutes I and $F = 0$. It should be noted that in the 17th century this problem was presented as a one-variable and was solved by Bernouly Brothers. Of course at the time there was no discussion of Control problem. Mention was made of Control in the 1940s. Later on Euler solved it thoroughly. Euler proved that the Optimal Path can be obtained from the following First-order necessary condition^[6]:

$$\frac{\partial I}{\partial X} - \frac{d}{dt} \left(\frac{\partial I}{\partial \dot{X}} \right) = 0 \tag{8}$$

8 above is system of differential equations, 8 called Euler equation. Afterwards, Legendre proved its necessary condition. At first, it was thought to be both the sufficient and the necessary condition. However, it was later proved

to be the necessary condition. The second-order necessary condition put forward by Legendre is as follows:

$$\frac{\partial^2 \dot{X}}{\partial Z^2} \leq 0 \quad (9)$$

That is, if Matrixes 9 is Negative Definite or Semi Negative Definite, then the optimal path obtained from equation 8 would maximize objective Functional 7.

5- Solving the C.P. by Dynamic Programming

Richard Bellman^[4] solved Bolza C.P. 5 by means of the Principle of Optimality Based on this Principle, Bellman^[7] Came up with the following equation known as Bellman Equation^[8]:

$$-\frac{\partial J^*}{\partial t} = \text{Max}_{u(t)} [I(X, U, t) + \frac{\partial J^*}{\partial X} f(X, U, t)] \quad (10)$$

Boundary conditions to Bellman Equation are as follows:

$$J^*(X(t_1), t_1) = F(X_1, t_1)$$

This Relation is a partial differential equation in a n dimension space. It is often difficult to solve this equation. That is why economists and management experts do not make use of this method very often.

6- Solving C.P. by using the Maximum Principle

The Maximum Principle presented by Pontryagin^[5] is a good approach to solve control problems. In this approach the co-state variable (Y) is first mentioned and then the Hamilton Function considered as follows^[9]:

$$\text{Max}_U H = I(X, U, t) + Y f(X, U, t) \quad (11)$$

Then we get maximum of this function with respect to U :

$$\left\{ \begin{array}{l} \frac{\partial H}{\partial U} = 0 \\ \dot{Y} = -\frac{\partial H}{\partial X} \\ \dot{X} = \frac{\partial H}{\partial Y} \\ Y(t_1) = \frac{\partial F}{\partial X_1} \end{array} \right. \quad t_0 \leq t \leq t_1 \quad (12)$$

The above relations are a system of differential equations through which the Optimal control paths of U^* and the state variable of X^* are obtained.

7- C.G. Evance's First Article in Economics and Management

C.G. Evance is of the first economists who used Dynamic Programming to maximize the profit of a firm. He published an article titled The Dynamics of Monopoly^[1] in the Monthly Math journal in 1924. He included this article in his book titled Mathematical Introduction to Economics published in 1930.^[10]

Evance, in this article, refers to a monopolist who produces a single commodity in the amount of $Q(t)$ at any point in time and intends to maximize the profit of his firm, so he takes $Q(t)$ as a continuous function of time.

The monopolist's total cost function would be:

$$C = \alpha Q^2 + \beta Q + \gamma \quad (13)$$

On the other hand, he has no storage and the output of Q equals to the quantity demanded. The quantity of the demand is a function of $P(t)$, the price of the product, and also a function of rate of change of price, $P'(t)$. That means:

$$Q = a - bP(t) + hP'(t) \quad a, b > 0 \quad h \neq 0 \quad (14)$$

P is a continuous function of, t , and has derivative. The profit of the firm is:

$$\pi = PQ - C$$

π is the firm's profit at any point, t , and the total profit from $t = 0$ to $t = T$ would be^[11]:

$$\Pi = \int_0^T \pi(P, P') dt$$

The problem is to determine the optimal price path $P^*(t)$ in , $t \in [0, T]$, so that total profit of the firm would be maximized.

$$\text{Max } \Pi = \int_0^T \pi (P, P') dt \tag{15}$$

$$P(0) = P_0, P(T) = P_T$$

P_0 and P_T are given.

This is the Lagrange C.P. Using expressions 13, 14, we determine the amount of 15 and use 8, i.e the Euler Equation, to solve the problem and we have the linear second order differential equation as below^[12]:

$$P'' - \frac{b(1 + \alpha b)}{\alpha h^2} \cdot P = - \frac{a + 2\alpha a b + \beta b}{2\alpha h^2} \tag{16}$$

The solution is as follows^[12]:

$$P^*(t) = A_1 C^{rt} + A_2 e^{-rt} + \bar{P} \tag{17}$$

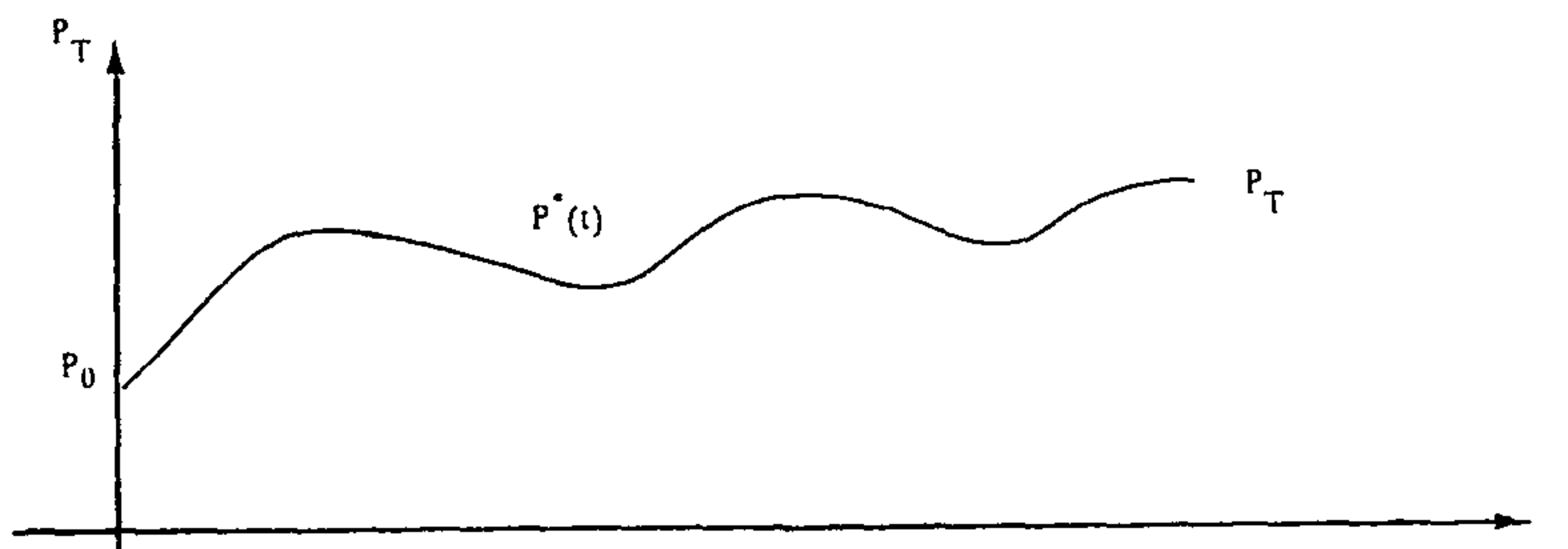
in which we would have:

$$r_1 + r_2 = \pm \sqrt{\frac{b(1 + \alpha b)}{\alpha h^2}}, \quad \bar{P} = \frac{a + 2\alpha a b + \beta b}{2b(1 + \alpha b)}$$

$$A_1 = \frac{P_0 - \bar{P} - (P_T - \bar{P}) e^{-rt}}{1 - e^{2rt}}, \quad A_2 = \frac{P_0 - \bar{P} - (P_T - \bar{P}) e^{-rt}}{1 - e^{-2rt}}$$

Using the legandre condition we can prove that $P(t)$ maximizes the profit in the time Interval $[0, T]$ ^[13]. Figure 3 shows the optimal price path $P^*(t)$ that Maximize the total profit Π over a finite period $[0, T]$.

Figure 3:



The example above is a simple case of Dynamic Programming. From those days until 1960 economists and management experts did make use of C.P. However, from the 1970s more and more use was made of C.P. This trend is on the go just now too.

8- Optimal Growth

The base of optimal growth is a subject that economists have used in C.P. very often. The main base for discussing this optimal growth is Dynamic Systems and C.P. In fact, this subject dealt with by a boundary value problem which was formulated by Ramsey in optimizing the saving problem in 1928^[2] and was Developed in 1956^[15] by Samuelson^[16] and Solow^[14]. Many articles have been published about it since the 1960s. Uzawa reaches the following control problem in his article titled Optimum Technological Change: an Aggregative Model of Economic Growth^[17]:

$$\begin{aligned} \text{Max } J &= \int_0^{\infty} [1-r(t)]y(t) e^{-\rho t} dt \\ \text{s.t.} & \\ & \bullet \dot{k}(t) = r(t) y(t) - \lambda k(t) \\ & \bullet \dot{A}(t) = r(t)\phi (1 - u(t)) \end{aligned} \tag{18}$$

Where $Y(t)$, is aggregate production function at each moment of time, t , and the state of Technological knowledge at time, t , is represented by the efficiency in labor, $A(t)$, it means $A(t)$ is labor efficiency, $y(t) = \frac{Y(t)}{L(t)}$ out

put per capita, $k(t) = \frac{K(t)}{L(t)}$ aggregate capita labor ratio, $u(t) = \frac{L_p(t)}{L(t)}$ labor

allocation productive sector, $r(t) = \frac{I(t)}{Y(t)}$ investment ratio, and ρ is the interest rate.

Uzawa has solved this problem using the Maximum Principle and found the optimal growth achieved by allocating labor and annual out put such that rate

of increase labor efficiency $\frac{\dot{A}}{A}$ equal the rate of increase the capital labor ratio,

$$\frac{\dot{k}}{k}$$

In the year 1990 Paul M. Romer in an article called endogenous Technical change discusses the optimal growth problem. He divides knowledge into 2 major components, Human Capital and Technology. He calls Human Capital as s_Y and Technology as A . If all the capital human is S_0 , S_A would be the capital human caused by education and research, and s_Y covers the rest; thus $S_Y + S_A = S_0$.

The rate of change of technology over time is $\frac{\dot{A}}{A} = \sigma S_A$. Romer, by viewing CP in terms of a function in the form of Bolza and solving it through the Maximum Principle. It follows that the parametrically expressed steady-state growth of Dynamic State is as follows:

$$\frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = \frac{\dot{C}}{C} = \frac{\dot{A}}{A} = \frac{\sigma (\alpha + \beta) S_0 - \alpha \rho}{\alpha \theta + \beta} = S_A \quad (19)$$

In which Y is the amount of production, K capital, C consumption, A technology, S_0 , all capital human, ρ rate of interest θ , β , α are constant coefficients. S_0 has a positive effect in growth, and σ , the parameter of success in research, also has a positive effect. In another article by Mercenier and Michel Philippe titled. Temporal Aggregation in Multi-sector Economy with Endogenous Growth^[19], published in the Journal of Economic Dynamic and Control in 2001, reference the Model is:

$$\text{Max } \int_0^{\infty} e^{-\rho t} h^\alpha(t) g[X(t), U(t)] dt$$

s.t.:

$$\begin{aligned} \dot{X}(t) &= f[X(t), U(t)], & X(0) &= X_0 \text{ given} \\ \frac{\dot{h}(t)}{h(t)} &= \phi[X(t), U(t)], & h(0) &= h_0 \text{ given} \end{aligned} \quad (20)$$

In the C.P. above $h(t)$ are Human Capital, $X(t)$ the n -dimension state variable, and $U(t)$ the control variable in m dimensions. This equation is simplified form of Lucas's model of growth is one of the most recent articles on the subject. In all these model extensive use has been made of C.P.

9- Stochastic Optimal Control

In some control problems we are faced with an uncertain state. For example, in such cases as facing a customer, the working time of a machine or the completion time of a project there is an uncertainty. In this state the state variable is not determined, but it has a random distribution. Instead of having, the motion equation: $\dot{X} = f(t, X, U)$, we have the formal stochastic differential equation

$$dx = f(t, X, U)dt + \sigma(t, X, U)dz \quad (21)$$

Where dz is increment of a "Stochastic process" z that obeys what is called "Wiener process". It is supposed for z and for a any partition t_0, t_1, t_2, \dots of time interval, the random variables $z(t_1) - z(t_0)$, $z(t_2) - z(t_1)$, $z(t_3) - z(t_2)$, ... are independent and has normal distribution with mean zero and variances $t_0 - t_1, t_2 - t_1, t_3 - t_2, \dots$ respectively. In this case the differentials elements of dt and dz would have the following multiplication table:

	dz	dt
dz	Dt	0
dt	0	0

Since $(dz)^2 = dt$ and Equation 10 (Bellman Equation) change to:

$$-\frac{\partial J^*}{\partial t} = \text{Max}_U \left(I(X, U, t) + \frac{\partial J}{\partial X} \cdot f(X, U, t) + \frac{1}{2} \sigma^2 \frac{\partial^2 J^*}{\partial^2 X} \right) \quad (22)$$

and the boundary condition remains, as before:

$$J(t_1, X(t_1)) = F(X_1, t_1)$$

To clarify the issue we refer to a model by Metron^[20], concerns allocating of personal wealth among current consumption, risk averse investment and riskul investment without transaction costs. This model is independent of time, Autonomous as follow^[21]:

$$\begin{aligned} \text{Max } J &= \int_0^\infty e^{-\rho t} \frac{c^b}{b} dt \\ \text{s.t.} \quad dw &= [s(1-w)W + \alpha w W - c] dt + w W \sigma dz \\ W(0) &= W_0 \end{aligned} \quad (23)$$

Where, W is the total wealth, w fraction of the wealth in the risky asset s return on the sure asset return on the risky asset, α expected efficiency of riskful investment, σ^2 the variance of return per unit time for riskful investment, c the consumption, $U(c) = \frac{c^b}{b}$ the utility function, $b < 1$ and ρ is the interest rate.

If we use formula 22, then the optimal amount would equal $J = AW^b$, where:

$$bA = \left[\frac{r - sb - (s - \alpha)^2 b}{2\sigma^2(1-b)} \right]^{b-1} (1-b)$$

thus c and w are equal to:

$$c = W(Ab)^{\frac{1}{b-1}} \quad w = \frac{\alpha - s}{(1-b)\sigma^2}$$

This branch of Dynamic Optimization grew a lot by Cumming & Shoartz^[22] from 1971 to 1977.

10 - Conclusions and Suggestions

Optimization is one of the most basic subjects in management and economics. Dynamic programming and Control problem are powerful tools in related problem analysis. In management and economic problems in which variables are continuous functions of time the optimization techniques are used. Students need to know differential and partial differential equations in order to apply these tools. Students of economics become familiar with these important subjects of mathematics. The students of managements will benefit from these methods if they become familiar with these mathematical topics. Papers presented in Iran very seldom consider these methods. It is hoped that more attention will be made to these topics in Ms or Ph.D programs.

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- 16- To get information about the Linear Differential Equation, refer to calculus, Volume 2, Chapter 10 (6th edition), Mohammad Hossein Pour Kazemi, Nashre Ney press.
- 17- To get more information see reference [15] chapters 3 and 4 titled, "Dynamic Systems" and "Control Problem and its Application in economics" respectively.
- 18- To prove this, see reference [6] chapter 10 page 284.
- 19- To prove this, see reference [6] chapter 9 page 262.
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