Iranian Int. J. Sci. ***6****(2), 2005, p.215-225*

An Extended Model of Asset Price Dynamics

M. H. Nojumi
Department of Mathematical Sciences, Sharif University of Technology
P.O. Box 11365-9415, Tehran, Iran. e-mail: nojumi@sina.sharif.edu

(*received: 17/5/2005 ; accepted: 5/10/2005*)

Abstract

An extended model of asset price dynamics for modeling stochastic upward and downward jumps in asset prices is developed, and the modified Black-Scholes solution for value of vanilla options is derived. The change in volatility is identified in detail using the Itô integrals and Itô formulas.

Keywords*: Stochastic jump, price dynamics, option pricing, volatility, Black-Scholes Analysis, vanilla options*

1. Introduction

A great variety of financial products are offered and traded in financial markets, of which options are one of the most frequently traded. A **call option** is a contract between two parties by which the buyer of the option (also called the holder of the option) has the *option*, not an obligation, to buy a specified (by quantity and quality) asset for a specified price at specified dates. If he chooses to do so, then the seller of the option (also called the writer of the option) has the *obligation* to sell the asset. A put option is a similar contract with the difference that the buyer of the option has the option to sell the asset, and if he chooses to do so, then the seller of the option has the obligation to buy the asset.

In the European option, the right to buy or sell can be exercised only at the expiration time of the option. In American option, this right can be exercised at any time till expiry. In practice, most options are of Bermudan type in which the right to buy or sell can be exercises only at certain pre-specified dates.

Hence, an option puts the holder in an advantaged position with respect to the writer. This advantaged position has money value. Hence, to become the holder of an option, one must pay some premium to purchase the option from the writer. The problem of option pricing is to determine how much premium one should pay to purchase an option; in other words, what the current price of the option is, given its parameters (asset price movement, exercise dates, exercise prices, etc.).

The European option is the simplest option to analyze and closed-form formulas are available for its value. Although in practice options are rarely European, the results for the European option are valuable since they can be generalized to obtain results for values of other types of options (Hull, 1997; Neftci, 1996; Wilmott *et al.,* 1995).

The underlying asset can be anything. One can consider an "asset", including cash, bonds, stocks, interest rates, foreign currency exchange rates, another option, various agricultural commodities like wheat or coffee, even things like weather condition, result of sports matches, etc. (Hull, 1997; Kohlmann and Tang, 2001). An important feature of asset prices is their *stochastic* movements, which in turn make prices of financial derivatives stochastic. Some assets display even fractal-like highly stochastic behavior (Peter, 1994).

2. Incorporation of Jump

Variations in an asset price are of two general types: expected and stochastic (Neftci, 1996). The Black-Scholes analysis assumes a log-normal *continuous* stochastic growth; described by the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t;$$

with $\{W_t : t \geq 0\}$ a Wiener process. A fundamental limitation of this model is that it does not capture the stochastic jumps in asset price; a phenomenon which happens in the real world. A stochastic process suitable for modeling rare events is the Poisson process (Ross, 2003). With $\{U_t : t \geq 0\}$ a Poisson process with rate λ which can be estimated by observing the market, the equation

$$dS_t = adt + \sigma dW_t + \gamma dU_t,$$

with a, σ and γ functions of S_t and t, models stochastic jumps as well but has two major problems:

- We are interested in capturing all average behavior of S_t in the deterministic term adt, and have the other two terms capture purely stochastic behavior of the asset price. The Wiener term adW_t has zero mean as desired. However, the Poisson term γdU_t does not have this feature since $U_{\Delta t}$ has mean $\lambda\Delta t \neq 0$.
- The Poisson counting process is non-decreasing. In the time interval $[t, t+\Delta t]$ the process U_t does not change with the probability $1-\lambda\Delta t + o(\Delta t)$, increases by one unit with the probability $\lambda\Delta t + o(\Delta t)$, and increases by more than one unit with the probability $o(\Delta t)$.

Hence with γ a nonnegative function, only the upward jumps in asset price are modeled. Although choosing a function γ which takes on both positive and negative values, downward jumps are also modeled, the first problem remains unresolved. The suggestion made here to simultaneously solve both the above problems is to introduce another Poisson process $\{D_t : t \geq 0\}$ with the same rate λ into the model in the following way:

$$dS_t = adt + \sigma dW_t + \gamma d(U_t - D_t);$$

with a, σ, and γ functions of S_t and t.

In every interval $[t, t+\Delta t]$, both $U_{\Delta t}$ and $D_{\Delta t}$ are Poisson random variables with mean $\lambda\Delta t$, hence $U_{\Delta t} - D_{\Delta t}$ has zero mean, as desired. The second issue is resolved as well even with γ a nonnegative function since in the time interval $[t, t+\Delta t]$ the process $U_{\Delta t} - D_{\Delta t}$

- stays the same with probability $\left[\lambda^2 + (1-\lambda)^2\right](\Delta t)^2 + o\left((\Delta t)^2\right)$,
- increases by one unit with probability $\lambda(1-\lambda)(\Delta t)^2 + o\left((\Delta t)^2\right)$,

- decreases by one unit with probability $\lambda(1-\lambda)(\Delta t)^2 + o\left((\Delta t)^2\right)$,
- increases by more than one unit or decreases by more than one unit with probability $o\left((\Delta t)^2\right)$.

3. Itô's Formula for Derivative Price

Ignoring all the $o(dt)$ terms, the differential of $V(S_t, t)$, the price of a financial derivative based on the underlying asset price S_t at time t, is

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S_t}dS_t + \frac{1}{2}\frac{\partial^2 V}{\partial S_t^2}dS_t^2.$$

In deterministic calculus, one would retain only the first two terms since all other terms are $o(dt)$. However, in stochastic calculus the third term cannot be ignored since it contains an $O(dt)$ term. To identify this term, we consider

$$\begin{aligned} dS_t^2 &= a^2 dt^2 + \sigma^2 dW_t^2 + \gamma^2 dJ_t^2 \\ &\quad + 2a\sigma dt dW_t + 2a\gamma dt dJ_t + 2\sigma\gamma dW_t dJ_t; \end{aligned}$$

with $J_t := U_t - D_t$. A fundamental property of the Wiener process is $dW_t^2 = dt$ (Neftci, 1996). To investigate the nature of dJ_t using the notion of the Itô integral, with an arbitrarily selected $T > 0$, we partition the time interval $[0, T]$ into n subintervals of equal length $h = T/n$ and, with $t_j := jh; j = 0, 1, ..., n$ and $\Delta J_j := J_{t_{j+1}} - J_{t_j}, j = 0, 1, ..., n-1$, we consider the quantity

$$\Omega_n := E\left(\left[\sum_{0 \le j \le n-1} \left(\Delta J_j\right)^2 - Z\right]^2\right);$$

with E the expectation operator, and Z the constant such that

$$\lim_{n\to 0}\Omega_n = 0.$$

Now

$$\Omega_n = E\left[\sum_{0\le j\le n-1}\left(\Delta J_j\right)^4 + Z^2\right.$$

$$\left. + 2\sum_{0\le j<k\le n-1}\left(\Delta J_j\right)^2\left(\Delta J_k\right)^2 - 2Z\sum_{0\le j\le n-1}\left(\Delta J_j\right)^2\right]$$

$$= \sum_{0\le j\le n-1} E\left[\left(\Delta J_j\right)^4\right] + Z^2$$

$$+ 2\sum_{0\le j<k\le n-1} E\left[\left(\Delta J_j\right)^2\left(\Delta J_k\right)^2\right] - 2Z\sum_{0\le j\le n-1} E\left[\left(\Delta J_j\right)^2\right].$$

By independent increments property of the Poisson process, for $j \ne k$ the random variables ΔJ_j and ΔJ_k are independent, so

$$E\left[\left(\Delta J_j\right)^2\left(\Delta J_k\right)^2\right] = E\left[\left(\Delta J_j\right)^2\right]E\left[\left(\Delta J_k\right)^2\right].$$

Moreover, by stationary increments property of the Poisson process, for every $j = 0,1,...,n-1$, the random variable

$$\Delta J_j := J_{t_{j+1}} - J_{t_j}$$

has the same probability distribution as the random variable

$$J_{t_{j+1-t_j}} = J_h \sim Poisson(\lambda h).$$

Hence

$$\Omega_n = \sum_{0 \le j \le n-1} E\left(J_h^4\right) + Z^2$$

$$+2 \sum_{0 \le j < k \le n-1} E\left(J_h^2\right)^2 - 2Z \sum_{0 \le j \le n-1} E\left(J_h^2\right)$$

$$= nE\left(J_h^4\right) + n(n-1)E\left(J_h^2\right)^2 - 2nZE\left(J_h^2\right) + Z^2.$$

Lemma. *The Moment generating function of a Poisson distribution with rate λ is*

$$\mathrm{MGF}_X(z) = \exp[\lambda\left(e^z - 1\right)].$$

Proof.

$$\mathrm{MGF}_X(z) = E\left(e^{zX}\right) = \sum_{k=0}^{\infty} e^{zk} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\left(\lambda e^z\right)^k}{k!}.$$

Having the moment generating function of J_h, we can determine the required quantities $E\left(J_h^2\right)$ and $E\left(J_h^4\right)$:

$$\frac{d\,\mathrm{MGF}_X}{dz}(z) = \lambda e^z \exp[\lambda\left(e^z - 1\right)]$$

$$\frac{d^2\,\mathrm{MGF}_X}{dz^2}(z) = \lambda e^z \left(1 + \lambda e^z\right) \exp[\lambda\left(e^z - 1\right)]$$

$$\frac{d^3\,\mathrm{MGF}_X}{dz^3}(z) = \lambda e^z \left(1 + 3\lambda e^z + \lambda^2 e^{2z}\right) \exp[\lambda\left(e^z - 1\right)]$$

$$\frac{d^4\,\mathrm{MGF}_X}{dz^4}(z) = \lambda e^z \left(1 + 7\lambda e^z + 6\lambda^2 e^{2z} + \lambda^3 e^{3z}\right) \exp[\lambda\left(e^z - 1\right)]$$

Since U_h and D_h are both Poisson processes with rate λh, we have

$$
\begin{aligned}
E\left(U_h\right) &= E\left(D_h\right) = \frac{d\,\mathrm{MGF}_{U_h}}{dz}(0) = \lambda h \\
E\left(U_h^2\right) &= E\left(D_h^2\right) = \frac{d^2\,\mathrm{MGF}_{U_h}}{dz^2}(0) = \lambda h(1+\lambda h) \\
E\left(U_h^3\right) &= E\left(D_h^3\right) = \frac{d^3\,\mathrm{MGF}_{U_h}}{dz^3}(0) = \lambda h\left(1+3\lambda h+\lambda^2 h^2\right) \\
E\left(U_h^4\right) &= E\left(D_h^4\right) = \frac{d^4\,\mathrm{MGF}_{U_h}}{dz^4}(0) \\
&= \lambda h\left(1+7\lambda h+6\lambda^2 h^2+\lambda^3 h^3\right)
\end{aligned}
$$

So

$$E\left(J_h\right)=E\left(U_h-D_h\right)=E\left(U_h\right)-E\left(D_h\right)=0;$$

$$
\begin{aligned}
E\left(J_h^2\right) &= E\left[\left(U_h-D_h\right)^2\right] \\
&= E\left(U_h^2\right)-2E\left(U_h\right)E\left(D_h\right)+E\left(D_h^2\right) \\
&= 2\left[\lambda h(1+\lambda h)-(\lambda h)^2\right] \\
&= 2\lambda h
\end{aligned}
$$

$$
\begin{aligned}
E\left(J_h^4\right) &= E\left[\left(U_h-D_h\right)^4\right] \\
&= E\left(U_h^4\right)-4E\left(U_h^3\right)E\left(D_h\right)+6E\left(U_h^2\right)E\left(D_h^2\right) \\
&\quad -4E\left(U_h\right)E\left(D_h^3\right)+E\left(D_h^4\right) \\
&= 2\lambda h\left(1+7\lambda h+6\lambda^2 h^2+\lambda^3 h^3\right) \\
&\quad -8\lambda h\left(1+3\lambda h+\lambda^2 h^2\right)+6\left[\lambda h(1+\lambda h)\right]^2 \\
&= 2\lambda h(1+6\lambda h)
\end{aligned}
$$

With these results, and by $\lambda h = T$, we obtain

$$\Omega_n = 2n\lambda h(1+6\lambda h) + 4n(n-1)\lambda^2 h^2 - 4n\lambda h Z + Z^2$$
$$= Z^2 - 4\lambda T Z + 4\lambda^2 T^2 + 2\lambda T + 8\lambda^2 T h$$

Therefore, noting that letting $n \to \infty$ is equivalent to letting $h \to 0$,

$$\lim_{n\to\infty} \Omega_n = Z^2 - 4\lambda T Z + 4\lambda^2 T^2 + 2\lambda T$$

The value of Z making this limit zero is

$$Z = 2\lambda T \pm i\sqrt{2\lambda T}$$

We have thus shown that

$$\int_0^T d\left(U_t - D_t\right)^2 = 2\lambda T \pm i\sqrt{2\lambda T}$$
$$= \int_0^T \left(2\lambda \pm i\sqrt{\frac{\lambda}{2t}}\right) dt.$$

For large t the imaginary term is negligible. We have thus proved

Proposition. *For large t compared to λ such that*

$$\sqrt{\frac{\lambda}{2t}} \ll 2\lambda \quad \text{or equivalently} \quad t \gg \frac{1}{8\lambda^2}$$

the variable $U_t - D_t$ satisfies the following equation with good approximation

$$d\left(U_h - D_h\right)^2 = 2\lambda dt.$$

Now we can identify the significant $O(dt)$ terms in dS_t^2:

$$
\begin{aligned}
dS_t^2 &= a^2 dt^2 + \sigma^2 dW_t^2 + \gamma^2 dJ_t^2 \\
&\quad + 2a\sigma dt dW_t + 2a\gamma dt dJ_t + 2\sigma\gamma dW_t dJ_t \\
&= \left(\sigma^2 + 2\lambda\gamma^2 + 2\sqrt{2\lambda}\rho\sigma\gamma\right) dt;
\end{aligned}
$$

with $-1 \le \rho \le 1$ the instantaneous coefficient of correlation between W_t and $U_t - D_t$, which here we define via the equation

$$
dW_t d(U_t - D_t) = \sqrt{2\lambda}\rho dt.
$$

It is reasonable to assume that under normal operation of the market, economic factors that contribute to upward stochastic jumps in price of a particular asset are independent from those responsible for downward stochastic jumps in the price of the same asset. Also, an economic factor is not likely to cause both an upward jump and a downward jump in the asset price. Hence it is reasonable to assume statistical independence of U_t and D_t

However, one may expect correlation between W_t and $U_t - D_t$. A continuous stochastic movement of the asset price may trigger factors that cause stochastic jumps in either direction. So both positive and negative values of the correlation coefficient ρ are likely. The Itô's formula for the financial derivative on the underlying asset price S_t then takes the form

$$
\begin{aligned}
dV &= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S_t} dS_t + \frac{1}{2}\frac{\partial^2 V}{\partial S_t^2} dS_t^2 \\
&= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S_t}\left(a dt + \sigma dW_t + \gamma d(U_t - D_t)\right) \\
&\quad + \frac{1}{2}\left(\sigma^2 + 2\lambda\gamma^2 + 2\sqrt{2\lambda}\rho\sigma\gamma\right)\frac{\partial^2 V}{\partial S_t^2} dt;
\end{aligned}
$$

or

$$dV = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\left(\sigma^2 + 2\lambda\gamma^2 + 2\sqrt{2\lambda}\rho\sigma\gamma \right) \frac{\partial^2 V}{\partial S_t^2} + a\frac{\partial V}{\partial S_t} \right) dt$$
$$+ \sigma \frac{\partial V}{\partial S_t} dW_t + \gamma \frac{\partial V}{\partial S_t} d(U_t - D_t).$$

4. The Updated Black-Scholes Equation

The Black-Scholes delta-hedged portfolio

$$\Pi := V - \frac{\partial V}{\partial S_t} S_t$$

is still risk-free since, keeping $\frac{\partial V}{\partial S_t}$ fixed during the infinitesimal time interval $[t, t+dt]$,

$$\begin{aligned} d\Pi &= dV - \frac{\partial V}{\partial S_t} dS_t \\ &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\left(\sigma^2 + 2\lambda\gamma^2 + 2\sqrt{2\lambda}\rho\sigma\gamma \right) \frac{\partial^2 V}{\partial S_t^2} + a\frac{\partial V}{\partial S_t} \right) dt \\ &\quad + \sigma \frac{\partial V}{\partial S_t} dW_t + \gamma \frac{\partial V}{\partial S_t} d(U_t - D_t) \\ &\quad - \frac{\partial V}{\partial S_t}\left(a dt + \sigma dW_t + \gamma d(U_t - D_t) \right) \end{aligned}$$

or

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\left(\sigma^2 + 2\lambda\gamma^2 + 2\sqrt{2\lambda}\rho\sigma\gamma \right) \frac{\partial^2 V}{\partial S_t^2} \right) dt.$$

In the absence of arbitrage opportunities, which is the usual case in the financial markets, the return from this portfolio should be equal to return from the risk-free investment of amount Π; that is, $r_t \Pi dt$, with r_t the instantaneous risk-free interest rate at time t. We thus arrive at the partial differential equation satisfied for the financial derivative price $V(S_t,t)$:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\left(\sigma(S_t,t)^2+2\lambda\gamma(S_t,t)^2+2\sqrt{2\lambda}\rho\sigma(S_t,t)\gamma(S_t,t)\right)\frac{\partial^2 V}{\partial S_t^2}+r_t S_t\frac{\partial V}{\partial S_t}-r_t V=0$$

In the special case of the asset price having constant log-normal expected growth rate, log-normal continuous stochastic change, and log-Poisson jump stochastic change; that is,

$$a(S_t,t)=\mu S_t, \qquad \sigma(S_t,t)=\sigma S_t, \qquad \gamma(S_t,t)=\gamma S_t$$

with μ, σ, and γ constants, the asset price dynamics follows the stochastic differential equation

$$dS_t=\mu S_t dt+\sigma S_t dW_t+\gamma S_t d(U_t-D_t)$$

With risk-free interest rate also assumed constant r, the partial differential equation satisfied by the financial derivative price $V(S_t,t)$ is

$$\frac{\partial V}{\partial t}+\frac{1}{2}\left(\sigma^2+2\lambda\gamma^2+2\sqrt{2\lambda}\rho\sigma\gamma\right)S_t^2\frac{\partial^2 V}{\partial S_t^2}+rS_t\frac{\partial V}{\partial S_t}-rV=0.$$

This is the Black-Scholes equation with σ replaced by the *effective* volatility $\hat{\sigma}$ given by

$$\hat{\sigma}^2:=\sigma^2+2\lambda\gamma^2+2\sqrt{2\lambda}\rho\sigma\gamma;$$

Three special cases are

- $\hat{\sigma}=\sigma+\sqrt{2\lambda}\gamma$ in case of full positive correlation ($\rho=-1$)
- $\hat{\sigma}=\sigma-\sqrt{2\lambda}\gamma$ in case of full negative correlation ($\rho=-1$)
- $\hat{\sigma}=\sqrt{\sigma^2+2\lambda\gamma^2}$ in case of no correlation ($\rho=0$).

A stock that has price S_t at time t and pays a continuous dividend yield q behaves like a non-dividend paying stock which has price

$e^{-q(T-t)}S_t$ at time t. Hence the price of European call and put options on this stock are given by the Black-Scholes formulas (Hull, 1997; Neftci, 1996) with σ replaced by $\hat{\sigma}$; that is,

$$C(S_t,t) = e^{-q(T-t)}S_t N(d_1) - e^{-r(T-t)}KN(d_2)$$

and

$$P(S_t,t) = e^{-r(T-t)}KN(-d_2) - e^{-q(T_t)}S_t N(-d_1)$$

with

$$d_1 := \frac{1}{\hat{\sigma}\sqrt{T-t}}\left[\log\left(\frac{S_t}{K}\right) + \left(r - q + \frac{1}{2}\hat{\sigma}^2\right)(T-t)\right],$$

$$\begin{aligned} d_2 &:= \frac{1}{\hat{\sigma}\sqrt{T-t}}\left[\log\left(\frac{S_t}{K}\right) + \left(r - q - \frac{1}{2}\hat{\sigma}^2\right)(T-t)\right] \\ &= d_1 - \hat{\sigma}\sqrt{T-t} \end{aligned}$$

Here T and K are the expiration time and exercise price of the option, respectively, and N is the cumulative distribution function of the standard normal distribution:

$$N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-s^2/2}ds.$$

The price of a European call or put option on a stock index, like S&P 500, can be obtained by setting as q the continuous return rate from that stock index. Similarly, the price of a European call or put option on a foreign currency can be obtained by replacing every q in the above formulas by the risk-free interest rate in the corresponding foreign country.

Acknowledgment

The author wishes to thank the Department of Mathematical Sciences and the Research Council of Sharif University of Technology for providing scientific and financial support.

References

Hull, J.C. (1997) *Options, Futures, and Other Derivatives*, Third Edition, Prentice Hall.

Kohlmann, M., and Tang, S. (2001) *Mathematical Finance, Proceedings of the Workshop on Mathematical Finance Research Project*, Konstanz, Germany, October 5-7, 2000, Birkhäuser Verlag.

Neftci, S. (1996) *An introduction to the mathematics of financial derivatives*, Academic Press, London.

Peter, E. (1994) *Fractal market analysis: applying chaos to the theory to investment and economics*, John Wiley & Sons, New York.

Ross, S. (2003) *Introduction to probability models*, Eighth Edition, Academic Press, London.

Wilmott, P., Howison, S., and Dewynne, J. (1995) *The mathematics of financial derivatives*, Cambridge University Press, Cambridge.