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On the Median-Unbiased Estimation of the Periodically Correlated AR(1) Coefficients

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Abstract

In this paper we consider the periodically correlated first-order autoregressive (PCAR(1)) process with period T and periodic white noise. One problem in studying this model is to estimate periodic coefficients from an observed segment. For ordinary stationary AR(1) model, a median unbiased estimate of the coefficient is well-known. This paper is concerned with the median-unbiased (MU) estimation of the periodic coefficients of the PCAR(1) process with period T. Our median unbiased estimator is an adaptation with the periodic case of the well-known work of Zielinski. The method of estimation is illustrated by simulated data.
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Keywords: *periodically correlated processes, AR(1) processes, median-unbiased estimation*

1. Introduction

In this work we consider periodically correlated first-order auto-regressive (PCAR(1)) process with period T,

$$X_t = \alpha_t X_{t-1} + \varepsilon_t , \qquad (1.1)$$

where $\alpha_{t+T} = \alpha_t$ and ε_t has the following properties

$$i) \qquad E\varepsilon_t = 0$$
$$ii) \qquad \sigma_{t+T}^2 = \sigma_t^2$$
$$\text{iii)} \quad \varepsilon_t \text{ are independent,}$$

for all $t \in Z$, with Z standing for all integers. The PCAR models are introduced by Jones and Brelsford (1967). Physical phenomena that involve periodicities give rise to random data for which appropriate probabilistic models exhibits periodically time-variant parameters, e.g., in the mechanical-vibration monitoring and diagnosis for machinery, atmospheric science, radioastronomy, biology, communications, telemetry, radar and sonar. For these and many other examples, the periodicity can be an important characteristic that should be reflected in an appropriate probabilistic model.

One problem in studying the model (1.1) is to estimate α_t from an observed segment $X_1, X_2, ..., X_n$ where n is fixed. For ordinary stationary AR(1) models, i.e., $X_t = \alpha X_{t-1} + \varepsilon_t$, $t \in Z$, Hurwicz (1950, p. 368) conjectured that the median of the ratio $\frac{X_t}{X_{t-1}}, t = 2,3,...,n$ would be a more efficient estimate of α and perhaps an unbiased one. Andrews (1993) found an exactly median-unbiased estimator of α. Both assumed that ε_t are i.i.d N (0, 1) random variables.

Zielinski (1999) proved the Hurwicz's conjecture whenever the medians of independent (not necessary identically distributed) innovations $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$ are equal to zero.

Our median-unbiased (MU) estimator of the parameters α_t, $t = 1,...,T$ from a PCAR(1) process is an adaptation with the periodic case of the work of Zielinski (1999).

2. The Generalized Hurwicz's Estimator

We assume that the innovations are independent, their medians are equal to zero and they are symmetric in the sense that $P(\varepsilon_t \geq 0) = P(\varepsilon_t \leq 0) = \frac{1}{2}$ for all $t = 1,2,..,n$ and $P(X_t = 0) = 0$ for $t = 1,2,..,n-1$. Also let $n = (k+1)T + 1$.

Based on Hurwicz's observation (1950) we conjecture that each of the ratios

$$\frac{X_t}{X_{t-1}}, \frac{X_{t+T}}{X_{t-1+T}}, \ldots, \frac{X_{t+kT}}{X_{t-1+kT}}, \tag{2.1}$$

are MU estimators of $\alpha_t, t = 2,3,\ldots T+1 \quad (\alpha_1 = \alpha_{T+1})$. If ε_t's are normally distributed, the ratio $\frac{X_t}{X_{t-1}}$ has a Cauchy distribution, and so it seems that the median of the suitable segment of the above ratios would be a MU estimator of α_t. In the following section we prove this conjecture.

3. Main Result

Consider each of the ratios $\frac{X_t}{X_{t-1}}, \frac{X_{t+T}}{X_{t-1+T}}, \ldots, \frac{X_{t+kT}}{X_{t-1+kT}}$ as an elementary MU estimator of α_t. Take

$$\hat{\alpha}_t = med\left(\frac{X_t}{X_{t-1}}, \frac{X_{t+T}}{X_{t-1+T}}, \ldots, \frac{X_{t+kT}}{X_{t-1+kT}}\right), \tag{3.1}$$

as Generalized Hurwicz MU (GHMU) estimator of α_t.We will show that $\hat{\alpha}_t$ is a median-unbiased estimator of α_t. The following lemma which is proved by Zielinski (1999) is useful in our work.

Lemma 3.1. (Zielinski (1999)). Let $U_1, U_2, \ldots, U_N$ be random variables such that

i) $P(U_i \leq c) = \frac{1}{2}$ for all $i = 1,2,\ldots,N$, c be a constant,

ii) for every $m = 1,2,\ldots,N$, for every choice $i_1, i_2, \ldots, i_m$ $(1 \leq i_1 < i_2 < \ldots < i_m \leq N$ of integers and for every $x_1, x_2, \ldots, x_{m-1}$,

$$P(U_{i_m} \leq c \mid U_{i_1} = x_1, \ldots, U_{i_{m-1}} = x_{m-1}) = \frac{1}{2},$$

then

$$P(med(U_1, U_2, ..., U_N) \le c) = \frac{1}{2}.$$

The following theorem is the main result of this paper.

Theorem 3.1 If the innovations $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$ are independent random variables such that $P(\varepsilon_t \ge 0) = P(\varepsilon_t \le 0) = \frac{1}{2}$ for all $t = 1,2,...,n$ and $P(X_t = 0) = 0$ for all $t = 1,2,...,n-1$, then $\hat{\alpha}_t$ given by (3.1) is median-unbiased: $P(\hat{\alpha}_t \ge \alpha_t) = \frac{1}{2}$ for all $t = 2,...,T+1$.

Proof. For the sequence (2.1) of the ratios, let $N = k$ and apply the lemma with

$$U_i = \frac{X_{t+(i-1)T}}{X_{t+(i-1)T-1}}, i = 1,..,k.$$

In this case the model (1.1) reduces to

$$X_{t+(i-1)T} = \alpha_t X_{t+(i-1)T-1} + \varepsilon_{t+(i-1)T},$$

and so

$$U_i = \alpha_t + \frac{\varepsilon_{t+(i-1)T}}{X_{t+(i-1)T-1}},$$

where $\varepsilon_{t+(i-1)T}$ and $X_{t+(i-1)T-1}$ are independent random variables. So

$$P_{\alpha_t}(U_i \le \alpha_t) = P_{\alpha_t}\left(\frac{\varepsilon_{t+(i-1)T}}{X_{t+(i-1)T-1}} \le 0\right)$$

$$= P_{\alpha_t}\left(\varepsilon_{t+(i-1)T} \le 0, X_{t+(i-1)T-1} > 0\right) + P_{\alpha_t}\left(\varepsilon_{t+(i-1)T} \ge 0, X_{t+(i-1)T-1} < 0\right)$$

$$= \frac{1}{2} P_{\alpha_t}\left(X_{t+(i-1)T-1} > 0\right) + \frac{1}{2} P_{\alpha_t}\left(X_{t+(i-1)T-1} < 0\right)$$

$$= \frac{1}{2}.$$

Therefore condition (i) of the lemma holds.

For every $m = 2,3,...,k$,for every choice of integers $i_1, i_2, ..., i_m$ $(1 \le i_1 < i_2 < ... < i_m \le k)$ and for every $x_1, x_2, ..., x_{m-1}$, , we observe that $\varepsilon_{t+(i_m-1)T}$ is independent of $X_i, i < t + (i_m - 1)T$. Therefore $\varepsilon_{t+(i_m-1)T}$ is independent of $U_{i_1}, U_{i_2}, ..., U_{i_{m-1}}$. We have

$$
\begin{aligned}
&P_{\alpha_t}(U_{i_m} \le \alpha_t \mid U_{i_1} = x_1, U_{i_2} = x_2, ..., U_{i_{m-1}} = x_{m-1}) \\
&= P_{\alpha_t}\left(\frac{\varepsilon_{t+(i_m-1)T}}{X_{t+(i_m-1)T-1}} \le 0 \mid U_{i_1} = x_1, U_{i_2} = x_2, ..., U_{i_{m-1}} = x_{m-1} \right) \\
&= P_{\alpha_t}\left(\varepsilon_{t+(i_m-1)T-1} \le 0, X_{t+(i_m-1)T-1} > 0 \mid U_{i_1} = x_1, U_{i_2} = x_2, ..., U_{i_{m-1}} = x_{m-1}\right) + \\
&P_{\alpha_t}\left(\varepsilon_{t+(i_m-1)T-1} > 0, X_{t+(i_m-1)T-1} < 0 \mid U_{i_1} = x_1, U_{i_2} = x_2, ..., U_{i_{m-1}} = x_{m-1}\right) \\
&= \frac{1}{2} P_{\alpha_t}\left(X_{t+(i_m-1)T-1} > 0 \mid U_{i_1} = x_1, U_{i_2} = x_2, ..., U_{i_{m-1}} = x_{m-1}\right) + \\
&\frac{1}{2} P_{\alpha_t}\left(X_{t+(i_m-1)T-1} < 0 \mid U_{i_1} = x_1, U_{i_2} = x_2, ..., U_{i_{m-1}} = x_{m-1}\right) \\
&= \frac{1}{2}.
\end{aligned}
$$

So that the second condition of the lemma is satisfied and the theorem follows, i.e. $\hat{\alpha}_t = med(\frac{X_t}{X_{t-1}}, \frac{X_{t+T}}{X_{t-1+T}}, ..., \frac{X_{t+kT}}{X_{t-1+kT}})$ is median-unbiased estimator for α_t.

4. Simulation

In this section we consider the periodic autoregressive model (1.1) with period T, where $\{\varepsilon_t\}$ is a sequence of independent random variables. First let $T = 2$ and

$$\varepsilon_t \sim \begin{cases} N(0,1), & \text{when } t \text{ is odd} \\ N(0,4), & \text{when } t \text{ is even} \end{cases} \quad , \quad \alpha_t = \begin{cases} 0.8, & \text{when } t \text{ is odd} \\ 0.1, & \text{when } t \text{ is even} \end{cases}$$

We compare this theoretical form with our estimates with generating $n = 1000$ observations from X_t. Using these observations we constructed 2-segment $\frac{x_{2l+1}}{x_{2l}}$ (for $l = 1,2,\ldots,499$) and $\frac{x_{2k}}{x_{2k-1}}$ (for $k = 1,2,\ldots,500$). The estimate $\hat{\alpha}_1$ is the median of $\frac{x_{2l+1}}{x_{2l}}$ ($l = 1,2,\ldots,499$) and $\hat{\alpha}_2$ is the median of $\frac{x_{2k}}{x_{2k-1}}$ ($k = 1,2,\ldots,500$). The estimates are given below

$$\hat{\alpha}_1 = .763268$$
$$\hat{\alpha}_2 = .114817,$$

with errors

$$|\alpha_1 - \hat{\alpha}_1| = .036702$$
$$|\alpha_2 - \hat{\alpha}_2| = .014817.$$

With using a similar method, some exact values of α_t with their estimates, $\hat{\alpha}_t$, for different values of T and n are given in Tables 1-4. We see the absolute values of the errors are negligible, which shows that the method of estimation is satisfactory.

Table 1- Exact values of α_t with estimates, $\hat{\alpha}_t$, $T = 2$ and $n = 400$.

dist. of ε_t	α_t	$\hat{\alpha}_t$	$\lvert\alpha_t - \hat{\alpha}_t\rvert$
$N(0,9)$	-0.4	-0.33795	0.06205
$N(0,16)$	-0.8	-0.83017	0.03017

Table 2- Exact values of α_t with estimates, $\hat{\alpha}_t$, $T=3$ and $n=1500$.

dist. of ε_t	α_t	$\hat{\alpha}_t$	$\lvert \alpha_t - \hat{\alpha}_t \rvert$
$N(0,9)$	0.3	0.35995	0.05995
$N(0,16)$	-0.7	-0.71454	0.01454
$N(0,2.25)$	0.5	0.47355	0.02645

Table 3- Exact values of α_t with estimates, $\hat{\alpha}_t$, $T=4$ and $n=2000$.

dist. of ε_t	α_t	$\hat{\alpha}_t$	$\lvert \alpha_t - \hat{\alpha}_t \rvert$
$N(0,1)$	-0.6	-0.62252	0.02252
$N(0,2.25)$	0.6	0.54395	0.05605
$N(0.4)$	0.4	0.38278	0.02722
$N(0,6.25)$	-0.9	-0.89061	0.00939

Table 4- Exact values of α_t with estimates, $\hat{\alpha}_t$, $T=4$ and $n=4000$.

dist. of ε_t	α_t	$\hat{\alpha}_t$	$\lvert \alpha_t - \hat{\alpha}_t \rvert$
$N(0,1)$	-0.6	-0.61153	0.01153
$N(0,2.25)$	0.6	0.54396	0.05604
$N(0.4)$	0.4	0.42856	0.02856
$N(0,6.25)$	-0.9	-0.85935	0.04065

5. An application

An illustrative PCAR(1) time series, discussed by Vecchia and Ballerini (1991), is the time series of mean monthly flows of Fraser River at Hope, BC, from January 1913 to December 1990. Mcleod (1994) showed that this time series exhibits periodic correlation and selected a PCAR(1) model, with periodicity of 12 months, $T=12$. The data is plotted in Fig. 1.

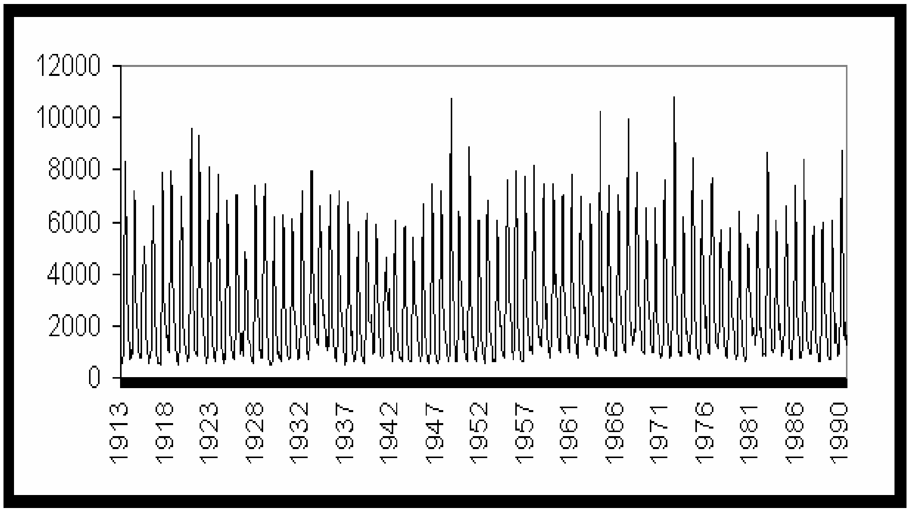

Fig 1. Mean monthly flows of Fraser River at Hope, BC, from January 1913 to December 1990.

In the following table, we have obtained the Generalized Hurwicz MU (GHMU) estimator of $\alpha_t, t = 1,...,12,$ for this set of data.

t	1	2	3	4	5	6
$\hat{\alpha}_t$	0.9503	1.9444	2.8955	1.4816	0.7811	0.6278

t	7	8	9	10	11	12
$\hat{\alpha}_t$	0.6601	0.8000	0.7636	0.6944	0.8431	0.9204

From these illustrations, we see that the estimators are satisfactory.

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