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More on θ-Compact Spaces

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Abstract

In this paper, we present and study the notion of firm θ-continuity to investigate θ-compactness.

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1. Introduction

In 1987, Noiri and Popa introduced and investigated a new class of functions called quasi θ-continuous functions. The second author of the present paper (Jafari, 1998) further investigated quasi θ-continuity and also introduced the notion of θ-compactness by using θ-open sets introduced by Velicko (1968). Kupka (1998) inspired by a number of characterizations of *UC* spaces (also called Atsuji spaces) (Waterhouse, 1965) to characterize compact spaces. For this purpose, he asked the question that what kind of continuity should replace uniform to be sufficiently strong to characterize compact spaces. He was able to tackle this problem by introducing a new type of continuity called firm continuity by which he obtained several characterizations of compact spaces.

It is the aim of this paper to continue the work of Kupka (1998) and obtain some characterizations of θ-compact spaces. In this relation we introduce the notion of firm θ-continuity which is natural for θ-compact spaces.

In what follows we denote the interior and the closure of a subset A of a topological space (X, τ) by $Int(A)$ and $Cl(A)$, respectively. A point $x \in X$ is called a θ-*adherent point* of a subset A of X if $A \cap Cl(V) \neq \emptyset$ for every open set V containing x. The set of θ-adherent points of A

is called the θ-*closure* of A which is denoted by $Cl_\theta(A)$. A subset A of X is called θ-*closed* if $A = Cl_\theta(A)$. The complement of a θ-closed set is called a θ-*open* set. A topological space (X, τ) is said to be θ-*compact* (Jafari, 1998) if every cover of X by θ-open sets has a finite subcover. A subset A of a topological space (X, τ) is said to be θ-*compact relative to* τ (Jafari, 1998) if for any cover $\{V_i \mid i \in I\}$ of A by θ-open sets of (X, τ) there exists a finite subset I_0 of I such that $A \subset \bigcup\{V_i \mid i \in I_0\}$. A topological space (X, τ) is called θ-T_1 (Jafari, 1998) if for every pair of distinct points x and y in X, there exists a θ-open set containing one of the points but not the other. A function is said to be *quasi* θ-*continuous* (Noiri and Popa, 1987) if every inverse image of each θ-open set in the codomain is θ-open in the domain.

2. Characterizations of θ-compact spaces

Definition 1: A function $f : X \to Y$, where X and Y are topological spaces, is said to have *property* ⎝ if for every θ-open cover ∇ of Y there exists a finite cover (the members of which need not be necessarily θ-open) $\{A_1, A_2, \ldots, A_n\}$ of X such that for each $i \in \{1, 2, \ldots, n\}$, there exists a set $U_i \in \nabla$ such that $f(A_i) \subset U_i$.

Lemma 2.1: A topological space X is θ-compact if and only if for every topological space Y and every quasi θ-continuous function $f : X \to Y$, f has the property θ.

Proof. Let the topological space X be θ-compact and the function $f: X \to Y$ be quasi θ-continuous. Suppose that Ξ is a θ-open cover of Y. The set $f(X)$ is θ-compact relative to Y. This means that there exists a finite subfamily $\{U_1, U_2, \ldots, U_n\}$ of Ξ which covers $f(X)$. Then the sets $A_1 = f^{-1}(U_1), A_2 = f^{-1}(U_2), \ldots, A_n = f^{-1}(U_n)$ form a cover of X such that $f(A_i) \subset U_1$ for each $i \in \{1, 2, \ldots, n\}$.

Conversely, suppose that X is a topological space such that for every topological space Y and every quasi θ-continuous function $f : X \to Y$, f has property θ. It follows that the identity function $id_X: X \to X$

has also property ⎩. Hence, for every θ-open cover Ξ of X, there exists a finite cover $A_1, A_2, \ldots, A_n$ of X such that for each $i \in \{1, 2, \ldots, n\}$ there exists a set $U_i \in \Xi$ such that $A_i = id_X(A_i) \subset U_i$. Then $U_1, U_2, \ldots, U_n$ are finite θ-subcover of Ξ. Since Ξ was an arbitrary θ-open cover of X, the space X is θ-compact.

Now we introduce the new class of firmly θ-continuous functions similar to the class of firmly continuous functions defined in (Kupka, 1998).

Definition 2: A function $f : X \to Y$ is called *firmly θ-continuous* if for every θ-open cover ∇ of Y there exists a finite θ-open cover Ξ of X such that for every $U \in \Xi$, there exists a set $G \in \nabla$ such that $f(U) \subset G$.

Remark 2.2: It should be noted that if the topological space X is θ-compact and Y is an arbitrary topological space, then every quasi θ-continuous function $f : X \to Y$ is firmly θ-continuous.

Lemma 2.3: Let X, Y, Z and W be topological spaces. Let $g : X \to Y$ and $h : Z \to W$ be quasi θ-continuous functions and let $f : Y \to Z$ be firmly θ-continuous. Then the functions $f \circ g : X \to Z$ and $h \circ f : Y \to W$ are firmly θ-continuous.

Lemma 2.4: Let X and Y be topological spaces. Suppose that $f : X \to Y$ is a quasi θ-continuous function which has the property ⎩. Then f is firmly θ-continuous.

Theorem 2.5: The following statements are equivalent for a topological space (X, τ):

(1) X is θ-compact;

(2) The identity function $id_X : X \to X$ is firmly θ-continuous;

(3) Every quasi θ-continuous function from X to X is firmly θ-continuous;

(4) Every quasi θ-continuous function from X to a topological space Y is firmly θ-continuous;

(5) Every quasi θ-continuous function from X to a topological space Y has the property ⎩;

(6) For each topological space Y and each quasi θ-continuous function $f : Y \to X$, f is firmly θ-continuous.

Proof. (1) $\Rightarrow$ (2) : Let X be θ-compact. The identity function id_X: $X \to X$ is quasi θ-continuous and by Remark 2.2 id_X is firmly θ-continuous.

(2) $\Rightarrow$ (3) : Let $f : X \to X$ be any quasi θ-continuous function. By (2), the identity function $id_X : X \to X$ is firmly θ-continuous. Therefore by Lemma 2.3 $f = id_X \circ f : X \to X$ is firmly θ-continuous.

(3) $\Rightarrow$ (4) : Suppose that $f : X \to Y$ is any quasi θ-continuous function. The identity $id_X: X \to X$ is quasi θ-continuous and by (3) id_X is firmly θ-continuous. As a consequence of Lemma 2.3, we have that $f = id_X \circ f : X \to Y$ is firmly θ-continuous.

(4) $\Rightarrow$ (5) : Obvious.

(5) $\Rightarrow$ (1) : This is an immediate consequence of Lemma 2.1.

(6) $\Rightarrow$ (2) : Suppose that $id_X: X \to X$ is the identity function. Then id_X is quasi θ-continuous and by (6) id_X is firmly θ-continuous.

(1) $\Rightarrow$ (6) : Suppose that ∇ is a θ-open cover of X. Since X is θ-compact, then there is a finite θ-subcover $U_1, U_2, \ldots, U_n$ of ∇. Let $A_i = f^{-1}(U_i)$ for $i \in I$, where $I = \{1, 2, \ldots, n\}$. We have that $f(A_i) \subset U_i$ for every $i \in I$. Therefore f is firmly θ-continuous.

Theorem 2.6: If $f : X \to Y$ is a firmly θ-continuous function, where X is a topological space and Y is a θ-T_1 topological space, then f is quasi θ-continuous.

Proof. Let x be an arbitrary point of X and V be a θ-open set of Y containing $f(x)$. We define a θ-open cover Ξ of Y such that $\Xi = \{V, Y - f(x)\}$. Since f is firmly θ-continuous, it follows that there exists a finite θ-open cover $\{P_1, P_2, \ldots, P_n\}$ of X such that $f(P_i) \subset V$ or $f(P_i) \subset Y - f(x)$ for every $i \in \{1, 2, \ldots, n\}$. Let $x \in P_j$ for some index j. Since $f(P_j)$ contains $f(x)$, so it follows that $f(P_j) \subset V$. This shows that *f* is quasi θ-continuous.

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