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Construction of some Join Spaces from Boolean Algebras

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Abstract

The aim of this paper is to construct an algebraic hyperstructure over a set G corresponding to a Boolean algebra B and a function $s{:}G \to B$. In order to accomplish this goal we will need to define a hyperoperation $\overset{s}{*}$ on the set G. We define,

$$a \overset{s}{*} b = \{ g \in G \mid s(g) \le s(a) \vee s(b) \}$$

and prove that if the image of G is a $\vee$-semilattice or constitute a partition of 1 in B, then (G, $\overset{s}{*}$) is a hypergroup.

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1. Introduction and Preliminaries

First of all we will recall some algebraic definitions that will be used in the paper. A hyperstructure is a set *H* together with a function *: H* ? *H* ? *P* (H)* called hyperoperation, where *P*(H)* denotes the set of all non-empty subsets of *H*. Marty (Marty,1934) defined a hypergroup as a hyperstructure (*H*,.) such that the following axioms hold: (i) *(x.y).z*?*x.(y.z)* for all *x.y, z* in *H*, (ii) *x.H=H.x=H* for all *x* in *H*. (ii) is called the reproduction axiom. A commutative hypergroup (*H*,*o*) is called a join space if for all *a,b,c,d* ?¢*H*, the implication *a/b* ?¢*c/d* ?¹0 ?¥ *aod* ?¥*boc* ?¹0 is valid, in which , *a/b*= $\{ x \mid a \in xob \}$.

The concept of an H_v-group is introduced by Vougiouklis in (Vougiouklis, 1994) and it is a hyperstructure *(H,.)* such that (i) *(x.y).z* ?° *x.(y.z)* ?¹0, for all *x, y,z* in *H*, (ii) *x.H=H.x* = *H* for all *x* in *H*. The first axiom is called weak associativity.

The partition function *p(n)* is defined as the number of sequences *(*a_1*,* a_2*, ... ,* a_r*), 0<*a_1?‡a_2?‡... ?a_r, that the positive integer *n* can be written as a sum of positive integers, a_i, as in $n=a_1+a_2+...+a_r$. The summands a_j are called the parts of the partition. Also, ? (*n)* will denote the set of all integer partitions of *n* and for every ?©?©?(*n*) we denote Part *(?)* the set of positive integers a_j such that $\sum_j a_j = n$.

In this paper we construct some join space from Boolean algebras. Our notations are standard and taken mainly from (Corsini, 1993) and (Vougiouklis, 1994).

2. Construction of some Join Spaces

Let *G* be a set, *B* a Boolean algebra and *s* be a function from *G* into *B*.

We define the hyperoperation $\overset{s}{*}$ as follows:

$$a \overset{s}{*} b = ?‡ ?\ G|s(x) ?\ s(a) ?\ s(b)?$$

Since for all *x, y* ?R*G*, $\{x,y\}$?R$x \overset{s}{*} y$, hence *(G,* $\overset{s}{*}$ *)* is an H_v-group. Also, it is obvious that the hyperoperation $\overset{s}{*}$ is commutative.

Example 1.

Suppose *G= ?Œ(6), I(6)=* = ?Œ,2,...,6?Œand *s: ?Œ(6)* ?Œ*P(I(6))* is defined by *s(?)=Part (?).* In Table I, we compute all of integer partitions of six. From this table we can see that :

(a $\overset{s}{*}$ *d)* $\overset{s}{*}$ *i* =?u*a,b,c,d,e,f,i*?u and *a* $\overset{s}{*}$ *(d* $\overset{s}{*}$ *i)=* ?u*a,b,c,d,e,i*?u

Therefore, *(a* $\overset{s}{*}$ *d)* $\overset{s}{*}$ *i*?*a* $\overset{s}{*}$ *(d* $\overset{s}{*}$ *i)* and so *(?/ (6),* $\overset{s}{*}$ *)* is not a hypergroup.

Some special cases where *(G,* $\overset{s}{*}$ *)* is a hypergroup are discussed in the following results.

Proposition 2.

If the image *G* is a ∨-sub-semilattice of *B* then *(G,* $\overset{s}{*}$ *)* is a commutative hypergroup.

Table I - Integer partitions of 6

a	6=1+1+1+1+1+1	b	6=1+1+1+1+2
c	6=1+1+2+2	d	6=2+2+2
e	6=1+1+1+3	f	6=1+2+3
g	6=1+1+4	h	6=1+5
i	6=3+3	j	6=2+4
k	6=6		

Proof.

Suppose *y* ?ú $(a \overset{s}{*} b) \overset{s}{*} c$, then there exists *g* ?ú *G* such that *s(g)* ?ƒ *s(a)* ∨ *s(b)* and *s(y)* ?ƒ *s(g)* ∨*s(c)*. Therefore, *s(y)* ?ƒ *(s(a)* ∨*s(b))*∨*s(c)=s(a)* ∨*(s(b)* ∨*s(c))*. Since the image *G* is a ∨-sub-semilattice of *B*, there exists *t* ?• *G* such that *s(b)* ∨*s(c)=s(t)* and so *s(y)* ?ñ*s(a)* ∨*s(t)*.Thus, *y* ?ñ $a \overset{s}{*} (b \overset{s}{*} c)$, i.e. $(a \overset{s}{*} b) \overset{s}{*} c$?ñ $a \overset{s}{*} (b \overset{s}{*} c)$. Similarly, we have $a \overset{s}{*} (b \overset{s}{*} c)$?Ν $(a \overset{s}{*} b) \overset{s}{*} c$. Therefore, the associative law is valid.

Lemma 3.

If the image *G* is a ?-sub-semilattice then we have,

$$a_1 \overset{s}{*} a_2 \overset{s}{*} \dots \overset{s}{*} a_n = \left\{ g \in G \middle| s(g) \le s(a_1) \vee \dots \vee s(a_n) \right\}.$$

Proof.

Suppose $U = a_1 \overset{s}{*} a_2 \overset{s}{*} \dots \overset{s}{*} a_n$ and

V ?‡?*g* ?†*G*|*s*(*g*) ?†*s*(a_1)?‡..?†*s*(a_n)?0

then we must prove *U=V*. It is easy to see that *U*?Ν*V*. Suppose *y*?Ν*V*, then *s(y)* ?€*s*(a_1) ∨...∨*s*(a_n). Since the image *G* is a ∨-sub-semilattice of *B*, hence there exists an element *g* ?‘ *G* such that *s(g)* = *s*(a_1) ∨…∨*s*(a_{n-1}). Using an inductive proof, we have *g*?] $a_1 \overset{s}{*} a_2 \overset{s}{*} \dots \overset{s}{*} a_{n-1}$ and *y*?]$g \overset{s}{*} a_n$. Therefore, *y*?]*U* and so *V* ?] *U*. This completes the proof.

Definition 4.

Let $B=(B, \vee, \wedge, 0, 1)$ be a Boolean algebra. A subset X? B is called a partition of *1* if and only if,

1) For all *x* ?X, *x*?0,

2) $1=\vee X$,

3) For all *x,y* ?X, *x*?*y*, we have $x \wedge y=0$.

For a Boolean algebra *B*, suppose A=Atom (*B*) is the set of all atoms of *B*. By 25. 1 and 2 of (Sikorski, 1964), if *B* is a complete Boolean algebra, *B* is a atomic if and only if it is completely distributive if and only if it is the field of all subsets of the set of all atoms of *B*. Therefore, the following lemma is true:

Lemma 5.

Let $B=(B, \vee, \wedge$,?,$0, 1)$ be an atomic complete Boolean algebra and A=Atom (*B*). An equivalence relation in the set *A* determines a partition of *1* and conversely, a partition of *1* defines an equivalence relation in *A*.

In fact the above lemma defines a one-to-one correspondence between the set of all partitions of *1* and the set of all equivalence relations on the set *A*.

Proposition 6.

If the image *G* is a partition of 1 then $(G, \overset{s}{*})$ is a commutative hypergroup.

Proof.

It is enough to show the associativity. Suppose *a, b, c* ?*G*,

$$(a \overset{s}{*} b) \overset{s}{*} c=\{g \in G \mid s(g) \leq s(a) \vee s(b)\} \overset{s}{*} c$$

$$= \bigcup_{s(g)\leq s(a)\vee s(b)} g \overset{s}{*} c$$

Set, $T=\{x \in G \mid s(x) \leq s(a) \vee s(b) \vee s(c)\}$. We now show that $T=(a \overset{s}{*} b) \overset{s}{*} c$. It is easy to see that $(a \overset{s}{*} b) \overset{s}{*} c$?*T*. Suppose *y* ?*T*, then *s(y)* ?*s(a)* $\vee$*s(b)* $\vee$*s(c)* and so $s(y)=(s(y) \wedge s(a)) \vee (s(y) \wedge s(b)) \vee (s(y) \wedge s(c))$. Now by hypothesis $\{s(g) \mid g \in G\}$ is a partition of 1 and our main proof will consider a number of cases.

Case 1) *s(y)=s(a)* or *s(y)* ?û*s(a)* and *s(y) =s(b)*. In this case we choose *g = y* and we have,

$$y \text{ ?ç } g \overset{s}{*} c \ \& \ s(g) = s(y) \text{ ?ç } s(a) \vee s(b).$$

Therefore, y ?Ç$(a \overset{s}{*} b) \overset{s}{*} c$.

Case 2) *s(y)* ?Ì*s(a), s(y)* ?Ì*s(b)* and *s(y) = s(c)*. In this case we choose *g=a* and we have,

$$y \text{ ?¸ } g \overset{s}{*} c \ \& \ s(g) = s(a) \text{ ?¸ } s(a) \vee s(b).$$

Thus, y ? $(a \overset{s}{*} b) \overset{s}{*} c$.

Case 3) *s(y)* ?ž*s(a), s(y)* ?ž*s(b)* and *s(y)* ?ž*(c)*. In this case we have *s(y)=0* and so y ?ú$(a \overset{s}{*} b) \overset{s}{*} c$. Similarly, $T=a \overset{s}{*} (b \overset{s}{*} c)$ and so $(a \overset{s}{*} b) \overset{s}{*} c=a \overset{s}{*} (b \overset{s}{*} c)$.

Lemma 7.

If the image *G* is a partition of 1 then we have,

$$a_1 \overset{s}{*} a_2 \overset{s}{*} \ldots \overset{s}{*} a_n=\left\{g \in G \middle| s(g) \le s(a_1)\vee \ldots \vee s(a_n)\right\}$$

Proof.

Suppose $a_1 \overset{s}{*} \ldots \overset{s}{*} a_{n-1}=\left\{g \middle| s(g) \le s(a_1)\vee \ldots . s(a_{n-1})\right\}$, then we have,

$$a_1 \overset{s}{*} \ldots . \overset{s}{*} a_n=\left\{g \middle| s(g) \le s(a_1)\vee \ldots . s(a_{n-1})\right\} \overset{s}{*} a_n$$

$$= \bigcup_{s(g)\le s(a_1)\vee \ldots \vee s(a_{n-1})} g \overset{s}{*} a_n$$

$$= \bigcup_{s(g)\le s(a_1)\vee \ldots \vee s(a_{n-1})} \left\{x \in G \middle| s(x) \le s(g) \vee s(a_n)\right\}$$

Set, $R=a_1 \overset{s}{*} \ldots \overset{s}{*} a_n$ and $S=\left\{g \in G \middle| s(g) \le s(a_1)\vee \ldots \vee s(a_n)\right\}$. It is obvious that *R* ?‡*S*, so it is enough to show that *S*?‡*R*. Suppose *x* is an arbitrary element of *S*, then *s(x)* ?™*s(a*$_1$*)* ?™...∨*s(a*$_n$*)*, and we have

$$s(x)=s(x) \wedge (s(a_1) \vee \ldots \vee (a_n)$$
$$= s(x) \wedge [s(a_1) \vee \ldots \vee (a_{n-1})] \vee [s(x) \wedge s(a_n)]$$

If $s(x)=s(a_n)$ then we choose $g = a_1$ and we have, $s(g)$? $s(a_1) \vee ... \vee s(a_{n-1})$, so $s(x)$? $s(g) \vee s(x)$? $s(a_1) \vee ... \vee s(a_n)$. We now assume that $s(x) \wedge s(a_n) = 0$, therefore $s(x)=s(x) \wedge (s(a_1) \vee ... \vee s(a_{n-1}))$. Choose $g=x$ and we have, $s(x)$? $s(g) \vee s(a_n)$, so $s(g)=s(x)$? $s(a_1) \vee ... \vee s(a_{n-1})$. This completes the proof.

Proposition 8.

If s is on to then $(G, \overset{s}{*})$ is a join space.

Proof.

Suppose s is onto, then by proposition 2, $(G, \overset{s}{*})$ is a hypergroup. Choose an element t such that $s(t)=0$, then t ? $a \overset{s}{*} d$? $b \overset{s}{*} c$, for all a, b, c, d ?_H. Therefore, $(G, \overset{s}{*})$ is a join space.

Lamma 9.

There exists a function s such that $(G, \overset{s}{*})$ is a hypergroup but it is not a join space.

Proof.

Suppose G is a Boolean algebra such that |Atom (G)| ?&4 and $s:G$?8 G defined by $s(0)=1$ and $s(x) = x$, for all x ?0. Since the image G is a $\vee$-sub-semilattice of G then by proposition 2, $(G, \overset{s}{*})$ is a hypergroup. We now assume that a,b,c,d are distinct atoms of G. It is clear that 1 ? a/b ? c/d and so a/b ?Ï c/d ?Ü0. But $a \overset{s}{*} d=$?ü$,d,a \vee d$?Ï and $b \overset{s}{*} c=$?ü$,c, b \vee c$?Ï If $a \vee d=b \vee c$ then $a = a \wedge (a \vee d) = a \wedge (b \vee c)=(a \wedge b) \vee (a \wedge c)=0 \vee 0=0$, a contradiction. Therefore, $(G, \overset{s}{*})$ is not a join space.

Proposition 10.

If the image G is a partition of 1 then $(G, \overset{s}{*})$ is a join space.

Proof.

Suppose the image of *G* is a partition of 1 and *a, b, c, d* ? *G*, such that *a/b* ?a*c/d*?a. *If s(a)=s(b)* then *a* ?a$a \overset{s}{*} d$?a$b \overset{s}{*}$ c and if *s(c)=s(d)* then *c* ?¾$a \overset{s}{*} d$?¾$b \overset{s}{*} c$. Therefore, we can assume that *s(a)*?¾*s(b)* and *s(c)* ?G*s(d)*. Now since the image *G* is a partition of *1*, hence $a/b=s^{-1}(s(a))$ and $c/d=s^{-1}(s(c))$. By assumption $s^{-1}(s(a)$?Ð$s^{-1}(s(c))$?Ð*0* and so *s(a) = s(c)*, i.e., a ?-$a \overset{s}{*} d$?- $b \overset{s}{*} c$, as required.

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