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Characterization of Certain Infinitely Divisible Distributions

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Abstract

The aim of this note is to characterize a class of distributions whose characteristic functions (t) for all $\alpha > 0$ satisfy the relation

$$[\,?(t\,)]^{?\ddagger} \;?\ddagger?(?\bullet^{H?\ddagger}t), \qquad ?\ddagger?\bullet R$$

where *H* and *H?*‹are positive real constants. First, it is observed that these distributions turn out to belong to the well-known class of infinitely divisible distributions. More specifically, they are strictly stable and thus absolutely continuous, self-decomposable and unimodal. But they are not strongly unimodal except for the case of *H=2H*. Then some representations are obtained for ?(*t)*. Finally, some characterizations are arrived at for the Cauchy and *N (0,* ?^{3}*); 0<* ?^{3} < ?Â distributions.

Key Words*: infinite divisibility, strictly stable distribution, Cauchy distribution, normal distribution, characteristics function, unimodality.*

1. Introduction

Recall that a random variable (r.v.) *X* with characteristic function (ch.f.) ?*(t)* is infinitely divisible (i.d.) if for every positive integer n we can write:

$$?(t) = [?_n(t)]^n, \quad t?\$R \tag{1}$$

where ?*(t)* is some ch.f. (see e.g., Feller, 1971). Further, the distribution is said to be stable if for every positive constants b_1 and b_2 there exists some positive constant b such that:

$$?(b_1 t)\ ?(b_2 t) = ?(bt) exp \{i?t\}, \quad ?\ t ?\ R \qquad (2)$$

where ? is some real constant. The distribution is said to be strictly stable if (2) holds with ?£=*0*. From Luckacs (1970, p137) if follows that, for every -∞ < *t* <+ ∞ , the ch.f. of a strictly stable distribution has the representation:

log?(*t*)=

$$\begin{cases} -c|t|^{?‡}[1+i?‡\frac{t}{|t|}\tan(??•/2)], & ?‡\ 1 \\ i?§-c|t| & ?‡=1 \end{cases} \qquad (3)$$

where *C*?0, |?|?1, *0*< ??2 and ? are real constants. The parameter γ is known as the characteristic exponent of the distribution.

In connection with self-similar compound stochastic processes, Alamatsaz and Lin (1997, Remark 3) encountered certain distributions whose ch.f ?(*t*) satisfy:

$$[?(t)]^{?‡'} \ ?‡?(?^{H'?‡}t),\ ?‡?\ 0, \quad t ?\ R \qquad (4)$$

Where *H* and *H?*½ are some positive constants. Clearly, well-known distributions such as the degenerate, normal with mean zero and Cauchy belong to this class of distributions.

In this note we shall study distributions whose ch.f.'s ?*(t)* satisfy relation (4). Section 2 reveals some structural properties of these distributions. It turns out that these distributions are not only i.d. but, more precisely, they are strictly stable and thus they are absolutely continuous, self-decomposable and unimodal However, they are not strongly unimodal except, for the case of *H=2H*?•In section 3, we shall give some representations for the ch.f's of such distributions. Finally, some characterizations are arrived at for *N (0,* ?^{2}*), 0*< ?^{2}< ?, and Cauchy distributions in section 4.

2. Structural properties

Despite of their odd look, as we see below, distributions defined in (4) are very well behaved from the structural point of view. Indeed, they have the following important properties.

(i) Closure under convolutions

Let X_1 and X_2 be two independent r.v.'s with ch.f.'s $\varphi_1(t)$ and $\varphi_2(t)$ respectively. Assume that φ_1 and φ_2 satisfy (4) with the same H and H' values. Then, $\varphi(t)$, the ch.f. of $Z=X_1+X_2$, also satisfies relation (4) with the same parameters. This is true because from (4) we can write:

$$\begin{aligned}\varphi(t) &= \varphi_1(t)\,\varphi_2(t)\\ &= [\varphi_1(\alpha^{H'}t)]^{\alpha^{-H}}\,[\varphi_2(\alpha^{H'}t)]^{\alpha^{-H}}\\ &= [\varphi(\alpha^{H'}t)]^{\alpha^{-H}},\end{aligned}$$

as required.

(ii) Infinite divisibility

Let $\varphi(t)$, the ch.f. of a r.v. X, satisfy (4). Then, since (4) is valid for every $\alpha > 0$, for every positive integer n we can write:

$$\begin{aligned}\varphi(t) &= [\varphi(n^{-\frac{H'}{H}}t)]^n \qquad t\in R\\ &= [\varphi_n(t)]^n \qquad t\in R\end{aligned}$$

where α is taken as $n^{-1/H}$ and $\varphi_n(t)$ is a ch.f. (indeed, the ch.f. of $Y=(n^{-\frac{H'}{H}})X)$. Thus, by (1), X is i.d

(iii) Strict stability

Let $\varphi(t)$, the ch.f. of a r.v. X, satisfy (4). First, we note that, by taking $\alpha = \beta^{-\frac{1}{H'}}$ in (4), for any $\beta > 0$ we have:

$$\begin{aligned}\varphi(\beta t) &= \varphi(\alpha^{-H'}t)\\ &= [\varphi(t)]^{\alpha^{-H}}\end{aligned}$$

$$?‡[\ ?[(t)]^{?‡} \qquad (5)$$

where $p = \frac{H}{H'}$. Thus, if b_1 and b_2 are two arbitrary positive constants, by (5), we obtain

$$?(b_1 t)?(b_2 t)\ ?‡[\ ?(t)]^{b_1^p}[\ ?(t)]^{b_2^p}$$
$$?‡?(bt),$$

where the constant $b = [b_1^p + b_2^p]^{\frac{1}{p}}$ is positive. Thus, by (2), X is strictly stable.

(iv) ***Other properties***

Distributions of type (4) are also self-decomposable. This is so because stable distributions are all self-decomposable (see, e.g, Gnedenko & Kolmogorov 1954, p.147). Therefore, by a result of Fisz & Varadarajan (1963) they are absolutely continuous and by Yamazato (1978) they are unimodal. However, they are not strongly unimodal in general. This follows from the fact that, according to Alamatsaz (1990), strongly unimodal distributions have finite moments of all orders. But, as seen from (3) or Theorems 3 and 4 below, distributions in question do not necessarily posses this property except when *H=2H?K* which leads to a normal distribution.

3. A representation

In the following theorem we give a simple representation for the ch.f. of a r.v. of type (4).

Theorem 1: *?(t)*, the ch.f of a r.v. *X*, satisfies relation (4) if, and only if,

$$?(t) = \begin{cases} A_1^{(?t)^p}, & t\ ?‡0 \\ A_2^{t^p}, & t\ ?<0 \end{cases}$$

where $p = \frac{H}{H'}$ and A_1 and A_2 are some constants (possibly complex).

Proof: If (6) is true, then obviously for any t < 0 we have

$$[?(t)]^{?H} ?[A_1^{(?)^p}]^{?H} ?sA_1^{?pH?(?t)^p} ?sA_1^{(??H?t)^p} ?s?(?s^{H?}t).$$

Hence, (4) is satisfied. The assertion is similar for $t ?y0$. Conversely, if (4) is hold, we may write:

$$[?(t)]^{?H} ??(?^{H?}t) ? ?(?^{H/p}t), \quad ?>0, \ t ? R \qquad (7)$$

It then follows from (7) that for all $t<0$

$$?(t) ??[(?)^{p/H}]^{H/p}(?) ?[?(?)]^{(?)^{p/H}?H} ?A_1^{(?t)^p}$$

with $A_1= ?(-1)$ and similarly for $t ?0$,

$$?(t) ??[(t^{p/H})^{H/p}] ?f[?(1)]^{t^{p/H}?H} ?f[?(1)]^{t^p} ?fA_1^{t^p}$$

with $A_2= ?(1)$. This completes the proof.

***Corollary 1*:**The ch.f. of a symmetric r.v., $?(t)$, satisfies (4) if, and only if,

$$?(t) = exp\{k?\}, \qquad t ?R \qquad (8)$$

where $p=\frac{H}{H'}>0$ and $k ?0$ are real constants.

Proof: The " if " part is obviously valid. Now, assume that X is symmetric and its ch.f. $?(t)$ satisfies (4). Then, the function $?(t)$ is real and even. Thus, $?(1) = ?(-1)=A$, where $A?$ is a real constant. Thus, by Theorem 1, we have

$$?(t)= A^{|t|^p}, \qquad t ?R, \qquad (9)$$

But since $?(0)=1$ and is continuous, there exists some $?0$ s.t. $?(s)>0$ for all ??. Hence, (9) implies that $A> 0$. Therefor, we have $0 < A ?1$ and

$$?(t)=exp\{k|t|^{p}\}, \quad t \; ?[R$$

with $k=logA$?Ð. As required.

Note: The result of above corollary is not surprising because, as seen before, the distributions given by (4) are infact strictly stable.

4. Characterization

First we give the following theorem.

Theorem 2: Let X be a r.v. with variance $0 < ?^{2} < ?$ Then, if its ch.f. $?(t)$ satisfies (4) we have $E(X)=0$ and $H=2H?$ (or equivalently $p=2$).

Proof: Since $?^{2} < ?$, $?(t)$ is twice differentiable. Differentiating from both sides of (4), we obtain

$$?^{H} ?(? t)[?(t)]^{?^{H}?} ?^{H?} ?(?^{H?} t), \quad ? \; 0, \; t \; ?R \qquad (10)$$

So at $t=0$, we have $?^{H} E(X) ? ?^{H?} E(X), ? ? 0$, or equivalently

$$(?^{H} ? ?^{H?})E(X) ? 0, \qquad ? \; 0, \qquad (11)$$

This obviously yields either $E(X)=0$ or $H=H?$ But, differentiating again from both sides of (10) we get:

$$?^{H} ?(t)[?(t)]^{?^{H}?} ?^{H}(?^{H} ?)[?(t)]^{2} ?^{2H?} ?(?^{H?} t), \quad ? \; 0, \; t \; ?R$$

At t=0, this yields,

$$(?^{H} ? ?^{2H?})E(X^{2}) ? ?^{H}(1 ? ?^{H})[E(X)]^{2} \qquad (12)$$

In view of (12), $H=H?$ implies that $E(X^{2})=E^{2}(X)$ and so $?^{2}=0$ which contradicts our assumption. Thus, by (11), we have $E(X)=0$ and hence by (12):

$$(\varepsilon^{H}-\varepsilon^{2H^{2}})\sigma^{2}=0 \qquad \forall\varepsilon>0$$

Since $\sigma^2>0$, it is therefore clear that $H=2H^2$ or equivalently $p=2$.

Theorem 3: Let $\varphi(t)$ be the ch.f. of a r.v. X with variance $0<\sigma^2<\infty$. Then, $\varphi(t)$ satisfies (4) if, and only if, X is $N(0,\sigma^2)$.

Proof: The "if" part is immediate. We only need to prove the "only if" part. Since $0<\sigma^2<\infty$, by Theorem 2 we have $p=2$. So in view of representation (3), Theorem 1 implies that

$$\log\varphi(t)=\begin{cases}-C(-t)^{p}\,[\,1+i\beta\tan(\pi p/2)\,], & t<0\\ -Ct^{p}\,[\,1-i\beta\tan(\pi p/2)\,], & t\ge 0\end{cases}$$

Thus, $A_1=\varphi(-1)=e^{-cB}$ and $A_2=\varphi(1)=e^{-c\bar{B}}$, where $B=1-i\beta\tan(\pi p/2)$. In this case, (6) gives.

$$\log\varphi(t)=\begin{cases}-ct^{2}B, & t<0\\ -ct^{2}\bar{B}, & t\ge 0\end{cases} \tag{13}$$

Differentiating twice, (13) yields:

$$\frac{d^2}{dt^2}\log\varphi(t)=\begin{cases}-2ct^{2}B, & t<0\\ -2ct^{2}\bar{B}, & t>0\end{cases} \tag{14}$$

Letting $t\to 0$, because $\sigma^2<\infty$, (14) implies $B=\bar{B}$ or $\beta=0$. Consequently, $c=\frac{1}{2}\sigma^2$. In this case, (13) reduces to

$$\log\varphi(t)=-\frac{1}{2}\sigma^2t^2, \qquad t\in R,$$

and therefore we have the result.

Finally, the theorem below easily follows from representation (3).

Theorem 4: Let ?*(t)*, the ch.f. of a r.v. *X*, satisfy (4). Then, *H=H?* if, and only if, X is a (general) Cauchy r.v.

In view of what we have seen above or more simply by a comparison of representations (3) and (8), it should be observed that the parameter $p=\frac{H}{H'}$ coincides with the characteristic exponent γ. Hence, we can conclude that in (4) we ought to have:

0 < H ?^2*H*?^

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