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Research Note

Estimating variance components of ellipsoidal, orthometric and geoidal heights through the GPS/levelling Network in Iran

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Abstract

The Best Quadratic Unbiased Estimation (BQUE) of variance components in the Gauss-Helmert model is used to combine adjustment of GPS/levelling and geoid to determine the individual variance components for each of the three height types. Through the research, different reasons for achievement of the negative variance components were discussed and a new modified version of the Best Quadratic Unbiased Non-negative Estimator (MBQUNE) was successfully developed and applied. This estimation could be useful for estimating the absolute accuracy level which can be achieved using the GPS/levelling method. A general MATLAB function is presented for numerical estimation of variance components by using the different parametric models. The modified BQUNE and developed software was successfully applied for estimating the variance components through the sample GPS/levelling network in Iran. In the following research, we used the 75 outlier free and well distributed GPS/levelling data. Three corrective surface models based on the 4, 5 and 7 parameter models were used through the combined adjustment of the GPS/levelling and geoidal heights. Using the 7-parameter model, the standard deviation indexes of the geoidal, geodetic and orthometric heights in Iran were estimated to be about 27, 39 and 35 cm, respectively.

Key words: Variance component estimation, Geoid, Levelling, GPS, BQUE, BQUNE, Iran

استفاده از روش آنالیز مؤلفه‌های واریانس در ارزیابی دقت مدل‌های ژئوئید جاذبی، بررسی موردی در ایران

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چکیده

ارزیابی درونی برای مدل ژئوئید جاذبی ایران IRG04 براساس قانون انتشار خطاها (جذر میانگین خطای مربعی عمومی مدل) دقتی معادل ۵/۲ سانتی‌متر را برآورد می‌کند. باید توجه داشت که دقت ارزیابی شده از این روش یک ارزیابی نظری و اغلب بسیار خوشبینانه است و با واقعیت‌های عملی انطباق ندارد. روش آنالیز مؤلفه‌های واریانس روشی متداول و مستقل در سرشکنی است. در این تحقیق از روش آنالیز مؤلفه‌های واریانس BQUE در قالب مدل سرشکنی ترکیبی گوس- هلمرت و به همراه اطلاعات نقاط ترازیابی- GPS و

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ژئوئید جاذبی در ایران شده است. نظر به اینکه این روش لزوماً همیشه برای مؤلفه‌های واریانس برآورد مقدار مثبت ندارد، روش جدید و توسعه یافته MBQUNE برای امکان تعیین مقادیر صرفاً مثبت در همه شرایط به لحاظ نظری طراحی شد. همچنین برنامه رایانه‌ای به‌زبان MATLAB برای آنالیز مدل‌های متفاوت سرشکنی پارامتریک به‌منظور تعیین مؤلفه‌های واریانس برنامه‌نویسی شد. این ارزیابی، اطلاعات ارزشمندی را در زمینه برآورد دقت‌های مدل ژئوئید جاذبی، ارتفاعات ارتومتریک و ارتفاع ژئودتیک ایران در بر داشت. نتیجه این تحقیق که براساس ۷۳ نقطه ترازیابی و GPS با توزیع مناسب در ایران و مدل ۷ پارامتری صورت پذیرفت، برآورد دقتی معادل ۷ سانتی‌متر را برای مدل ژئوئید جاذبی IRG04 داشت که این مقدار، انطباق بسیار خوبی با برآورد اولیه داخلی ما برای این مدل در ایران به‌دست می‌دهد.

مقدمه: روش آنالیز مؤلفه‌های واریانس (VCE) و خصوصاً روش MINQUE به‌صورت گسترده‌ای در ژئودزی کاربرد دارد. این روش را هلمرت (۱۹۰۷) به صورت مستقل ارائه کرد و پروفسور گرافارند (۱۹۸۵) و شوبرگ (۱۹۸۳ a,b) نقش بسزایی در توسعه آن داشته‌اند. روش VC قابلیت استفاده در مشاهدات همگن و ناهمگن را دارد و عموماً به منظور تغییر مقیاس واحد وزن مؤلفه‌های متفاوت در مدل‌های سرشکنی مورد استفاده قرار می‌گیرد. در این روش فرض بر این است توزیع خطاها در مشاهدات به‌صورت اتفاقی است. انگیزه‌های زیادی برای استفاده از روش VC در سرشکنی مدل ترکیبی ترازیابی/GPS و ژئوئید وجود دارد.

استفاده از این روش امکان درجه‌بندی کردن خطاهای مدل ژئوئید، ارتفاعات ژئودتیکی و ارتومتریک به‌صورت روشی کاملاً مستقل را فراهم می‌کند. مزیت این روش ترکیبی داشتن درجه آزادی زیاد در مدل VC است. به‌عبارت دیگر با وجود افزایش تعداد نقاط ترازیابی/GPS و ژئوئید، تعداد مؤلفه‌های مجهول ثابت است (سه مؤلفه). عموماً ماتریس واریانس و کواریانس مشاهدات مدل از نتایج سرشکنی شبکه‌های ترازیابی و GPS قابل استخراج است. در مورد دقت ژئوئید نیز براساس قانون انتشار خطاها (جذر میانگین خطای مربعی عمومی مدل) می‌توان برآورد مناسبی از دقت داخلی مدل داشت (رابطه ۱). این مقادیر درحکم داده‌های اولیه برای مدل VC مورد استفاده قرار می‌گیرند و از طریق مؤلفه‌های واریانس به روش تکرار به‌تدریج روزآمد می‌شوند. نکته مورد توجه در این روش این است که مؤلفه‌های واریانس همیشه برآورد مثبتی ندارند. این مسئله می‌تواند نشانگر وجود داده‌های اشتباه در مدل، خطاهای سیستماتیک و یا استفاده از مدل نامناسب در سرشکنی باشد. استفاده از روش BQUNE شوبرگ (۱۹۸۴) و اسحاق- شوبرگ (۲۰۰۸) می‌تواند روشی برای مقابله با مؤلفه‌های واریانس منفی باشد. با این حال بایستی قبل از استفاده از این روش، علت بروز آن را به دقت بررسی کرد.

مدل‌سازی مسئله: با استفاده از رابطه (۱) ارزیابی درونی برای مدل ژئوئید جاذبی ایران IRG04 براساس قانون انتشار خطاها (جذر میانگین خطای مربعی عمومی مدل) دقتی معادل ۵٫۲ سانتی‌متر را برآورد می‌کند.

$$\delta\bar{N}^2 = c^2 \sum_{n=2}^{n_{max}} \left[\left(\frac{2}{n-1} - s_n^* - Q_n^L \right)^2 \sigma_n^2 + \left(Q_n^L + s_n^* \right)^2 dc_n^* \right] \qquad (۱)$$

هرگاه $c = R/(2\gamma)$ ، $Q_n^L = Q_n - \sum_{n=2}^{\infty} \frac{2k+1}{2} s_k e_{nk}$ ، $s_n^* = \begin{cases} s_n & \text{if } 2 \le n \le L \\ 0 & \text{otherwise} \end{cases}$ ، Q_n نشانگر ضرایب ملودنسکی و s_n پارامترهای اصلاحی روش شوبرگ (۲۰۰۳ c) است.

همچنین متوسط خطای ارتفاعی نقاط GPS شبکه ایران حدود ۱۲٫۵ سانتی‌متر برآورد می‌شود (نانکلی- تماس شخصی) . از طرف دیگر توجه به نتایج شبکه ترازیابی ایران دقت متوسطی حدود ۷ سانتی‌متر را برآورد می‌کند (هامش، ۱۹۹۱ و NCC, 2003). این مقادیر در نقش تقریب اولیه به مدل سرشکنی ترکیبی ۷ پارامتری زیر معرفی شدند:

$$\boldsymbol{a}^T{}_i x = \begin{pmatrix} \cos\varphi_i \cos\lambda_i \\ \cos\varphi_i \sin\lambda_i \\ \sin\varphi_i \\ \cos\varphi_i \sin\varphi_i \cos\lambda_i / K_i \\ \cos\varphi_i \sin\varphi_i \sin\lambda_i / K_i \\ \sin^2\varphi_i / K_i \\ 1 \end{pmatrix}^T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} \qquad (۲)$$

φ و λ به‌ترتیب عرض و طول جغرافیایی نقاط هستند و e خروج از مرکز اولیه بیضوی مقایسه است. همچنین $K_i = \left(1-e^2 \sin^2 \varphi_i\right)^{1/2}$ است. ماتریس دوم طراحی برابر $B = [I_n \quad -I_n \quad -I_n]$ است که n تعداد نقاط ترازیابی و GPS را نشان می‌دهد. در این حال مدل اتفاقی VC را می‌توان به‌صورت زیر نوشت:

$$Q = \sigma_h \begin{bmatrix} Q_h & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \sigma_H \begin{bmatrix} 0 & 0 & 0 \\ 0 & Q_H & 0 \\ 0 & 0 & 0 \end{bmatrix} + \sigma_N \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Q_N \end{bmatrix} = \begin{bmatrix} \sigma_h Q_h & 0 & 0 \\ 0 & \sigma_H Q_H & 0 \\ 0 & 0 & \sigma_N Q_N \end{bmatrix} \qquad (۳)$$

بنابراین می‌توان نوشت:

$$BQB^T = \sigma_h Q_h + \sigma_H Q_{Hh} + \sigma_N Q_N = Q_W \qquad (۴)$$

برای تعیین مقادیر مجهول و باقی‌مانده‌ها می‌توان از معادله زیر بهره گرفت:

$$X = (A^T Q_w^{-1} A)^{-1} A^T Q_w^{-1} W \qquad (۵)$$

و داریم:

$$\varepsilon = QB^T Q_w^{-1} (I - A^0) W \qquad (۶)$$

هرگاه:

$$A^0 = AA^- = A(A^T Q_w^{-1} A)^{-1} A^T Q_w^{-1} \qquad (۷)$$

و مقادیر زیر قابل حصول

$$K_0 = (I - A^0) Q_w (I - A^0)^T = Q_w (I - A^0)^T \qquad (۸)$$

$$K_0^{-1} = (Q_w (I - A^0)^T)^{-1} = Q_w^{-1} \qquad (۹)$$

در نهایت مقادیر مؤلفه‌های VC به روش BQUE از راه معادلات زیر تعیین می‌شوند:

$$S_{ij} = \mathrm{trace}(Q_w^{-1} K_i Q_w^{-1} K_j) \qquad (۱۰)$$

$$q_i = \bar{W}^T Q_w^{-1} K_i Q_w^{-1} \bar{W} \qquad (۱۱)$$

نتیجه این تحقیق که براساس ۷۳ نقطه ترازیابی و GPS با توزیع مناسب در ایران (شکل ۲) و مدل ۷ پارامتری صورت پذیرفت، برآورد دقتی معادل ۷ سانتی‌متر را برای مدل ژئوئید جاذبی IRG04 داشت که این مقدار، انطباق بسیار خوبی را با برآورد اولیه داخلی ما برای این مدل در ایران ارائه می‌کند. در این تحقیق مدل‌های ۴ و ۵ پارامتری نیز مورد ارزیابی قرار گرفت. نتایج تحقیق بیانگر نامناسب بودن مدل ۴ پارامتری به خاطر برآورد منفی مؤلفه‌های واریانس بود. همچنین دقت ارزیابی شده برای ارتفاعات ژئودتیک و ارتومتریک در این شبکه به ترتیب برابر ۱۵ و ۱۲ سانتی‌متر است و روشن می‌سازد که ارزیابی اولیه برای این پارامترها اندکی خوشبینانه بوده است (جدول ۳). نمونه اجرای برنامه رایانه‌ای به همراه فلوچارت و مقادیر نمونه ورودی و خروجی در شکل‌های ۱ و ۳ ارائه شده‌اند.

واژهای کلیدی: آنالیز مؤلفه‌های واریانس، ژئوئید، ترازیابی، ایران، MBQUNE، BQUNE، BQUE، GPS، IRG04

1 INTRODUCTION

The method of variance component estimation (VCE) used in geodesy is essentially Rao's Minimum Norm Quadratic Unbiased Estimation- MINQUE (Rao, 1970, 1971, Rao and Kleffe, 1988). In the geodetic literature, the problem has been solved independently by Helmert (1907) and in recent times by Sjöberg (1983a, 1983b, 1984 and 1985). A review with further references to the literature can be found in Grafarend (1985).

The VCE can be applied to both homogeneous and heterogeneous sets of observables to determine the relative uncertainties of the various components that make up the total stochastic model. In the case of heterogeneous observations, each observational group may be assigned its own error characteristics. The VCE method uses an iterative numerical procedure starting with reasonable a priori estimates of the variance components. It relies on the assumption that all significant errors are randomly represented in the data set, and that a single scale factor is sufficient to account for the errors in one observation type, particularly where observations may seem consistent and yet still be affected by undetected systematic errors. This is usually true when dealing with the propagation of errors into new observation types such as the case in combination of the GPS/levelling and geoidal heights. Therefore, it is recommended that methods such as VCE be accompanied by attempts to evaluate the effects of systematic errors. However, there are different motivations for applying VCE techniques in combined adjustment of heterogeneous height data such as: It facilitates studies on the calibration of the geoid error model, the assessment of the noise in the heights derived from GPS measurements, for the evaluation of the levelling precision and provides an independent test of the error values associated with various orders of conventional spirit levelling (Fotopoulos, 2003 and Fotopoulos et al. 2003).

An interesting and advantageous property of the combined height adjustment problem is the inherent relatively high degree of freedom in solution. Although the distribution and number of GPS/levelling benchmarks vary, the number of unknown variance components (three in this case) remains the same (Fotopoulos, 2003). Thus, unlike many other VCE related applications, where the main obstacle encountered is the high computational load, the problem here lies in reliable variance estimates for each height type.

Basically, the empirical covariance matrices of the ellipsoidal, orthometric and geoidal heights (Q_h, Q_H and Q_N) can be computed from a-priori information about the accuracy of the three height types and used as initial input into the VCE algorithm. These a priori covariance matrices were successively 'updated' by the corresponding estimated variance components $\sigma^2 = \left[\sigma_h^2, \sigma_H^2, \sigma_N^2\right]$ in an iterative procedure. The iterations stop and the final estimated factors are computed once a pre-specified convergence criterion is met.

However, variance components may come out negative in some cases which have basically important meanings such as the presence of outliers, systematic error and modeling problems which need detailed investigations (Sjöberg, 1984).

One way to escape this problem is to use non-negative methods of variance components estimation. There exist some non-negative methods to estimate the variance components. One of these methods is Best Quadratic Unbiased Non-negative Estimation. This method has a shortcoming when it is used for condition or Gauss-Helmert model. In this article, it is shown how to face with this problem and modify it.

Also estimation of the variance components through the GPS/levelling networks was investigated in detail by Fotopoulos (2003) and Fotopoulos et al. (2003), but here we use an independent approach by utilizing the MATLAB function which always gives positive values for VC.

The article starts with a quick review of the application of Best Quadratic Unbiased Estimation of VC through the combined least-squares adjustment of Gauss-Helmert model. Then, the negative VC problem is discussed and the Modified Best Quadratic Unbiased Non-negative Estimator is introduced. Finally, a general MATLAB function is utilized for resolution of the problem in the positive and negative cases within a sample GPS/levelling network in Iran.

2 COMBINED LEAST SQUARES ADJUSTMENT OF THE GAUSS-HELMERT MODEL

Consider the following Gauss-Helmert model,

$$AX + B\varepsilon = W \tag{1}$$

where, **A** and **B** are the first and second design matrices, **X** is the unknown vectors and ε is the residual vector and **W** is the misclosure vector. For a solution of the Gauss-Helmert model we can referee to Sjöberg (1993) as:

$$X = (A^T(BQB^T)^{-1}A)^{-1}A^T(BQB^T)^{-1}W \tag{2}$$

and the residual vector will be

$$\varepsilon = QB^T(BQB^T)^{-1}(I - A^0)W \tag{3}$$

where, $A^0 = AA^-$ and A^- is the generalized inverse of the **A** according to Sjöberg (1993).

2-1 BEST QUADRATIC UNBIASED ESTIMATION OF VARIANCE COMPONENTS IN THE GAUSS-HELMERT MODEL

The model (1) can be converted to the classical condition adjustment model by multiplying both sides of the equations by $(I - A^0)$, namely

$$(I - A^0)B\varepsilon = (I - A^0)W, \tag{4}$$

The above model can be re-written as a condition model

$$\bar{B}\varepsilon = \bar{W}, \tag{5}$$

For the conditional adjustment, the best quadratic unbiased estimation of variance components can be written as:

$$\hat{\sigma} = S^- q, \tag{6}$$

where, S^- is the generalized inverse of S. It is obvious that it becomes S^{-1} when the S has full rank and $\hat{\sigma}$ is the variance-covariance components estimated in the Gauss-Helmert model (Sjöberg, 1993).

$$S_{ij} = \mathrm{trace}(K_0^{-1}K_iK_0^{-1}K_j), \tag{7}$$

$$q_i = \bar{W}^T K_0^{-1}K_iK_0^{-1}\bar{W}, \tag{8}$$

$$K_0 = \bar{B}Q_0\bar{B}^T = (I - A^0)BQ_0B^T(I - A^0)^T, \tag{9}$$

$$K_j = \bar{B}Q_i\bar{B}^T = (I - A^0)BQ_jB^T(I - A^0)^T, \tag{10}$$

2-2 MODIFIED BEST QUADRATIC UNBIASED NON-NEGATIVE ESTIMATION OF VARIANCE COMPONENTS

Estimation of the negative values for variance components is one of the main problems in VCE. It can be happened because of incomplete mathematical model or wrong stochastic model. Further, it can be cased by unsuitable a priori VCs, a few number of observations. In a problem with few numbers of variance components and a high degree of freedom, negative components seldom occur (Persson, 1980). But it is possible to use the non-negative methods too; one of these methods proposed by Sjöberg (1984) is the Best Quadratic Unbiased Non-negative Estimation of variance components. This method is briefly presented as follow:

In order to obtain the non-negative variance components we should transform the $\bar{W}$ to the

$$\gamma_i = F_i\bar{W}, \tag{12}$$

where we have chosen

$$F_i = I - C_0C_0^-, \text{ and } C_0 = C - \sigma_i^2 C_i, \tag{13}$$

where,

$$C = BQB^T, \text{ and } C_i = BQ_iB^T, \tag{14}$$

then γ_i is normally distributed with expectation 0 and

$$E\{\gamma_i\gamma_i^T\} = \sigma_i^2 F_iS_iF_i, \tag{15}$$

where

$$S_i = C_i(I - A^0)^T, \tag{16}$$

Hence, if $P_i = F_iS_iF_i \neq 0$, then BQUNE of σ_i^2 among the estimators $\gamma_i^T M\gamma_i$ is given by (Sjöberg, 2003a)

$$\hat{\sigma}_i^2 = \gamma_i^T P_i^- \gamma_i / \mathrm{rank}(P_i), \tag{17}$$

where, P_i^- is any generalized inverse of P_i. As we know, the Gauss-Helmert Model will be transformed to the famous Gauss-Markov model, when **B=I**, and it will be a condition model if **A=0**. Let us start our discussion with the Gauss-Markov model. If in the above relations we put **B=I**, all of the relations appear in the Gauss-Markov model as

$$AX = W + \varepsilon = L + \varepsilon, \tag{18}$$

subsequently, we have

$$\gamma_i = F_i(I - A^0)W = F_i\varepsilon, \tag{19}$$

where, **W** plays the role of L vector of observations. In the Gauss-Markov model C=Q, and $C_i = Q_i$, so F_i can easily be written as

$$F_i = I - Q_0Q_0^-, \text{ where } Q_0 = Q - \sigma_i^2 Q_i, \tag{20}$$

and also

$$S_i = Q_i(I - A^0)^T, \tag{21}$$

In this case, F_i plays the role of a separator. It separates the vector of residuals corresponding to each group of observations and puts it in γ_i. Finally the non-negative variance components can easily be computed using the Equation (8).

Now let us consider the condition model, if in the Gauss-Helmert model we suppose **A=0**, so the model becomes a condition model:

$$B\varepsilon = W, \quad (22)$$

In a similar manner, we can obtain the relations for computing the non-negative variance components as

$$\gamma_i = F_i W, \quad (23)$$

where

$$F_i = I - C_0 C_0^-, \text{ where } C_0 = C - \sigma_i^2 C_i, \quad (24)$$

and also

$$S_i = C_i, C = BQB^T \text{ and } C_i = BQ_i B^T, \quad (25)$$

In the case where C_0 becomes regular, F_i =0 and no solution exists. This problem occurs in calibration of precision of GPS/levelling and Geoid. Most of persons working on such a problem suggest using the BQUNE method when negative values come out, but we must mention that this method is not appropriate for use in the error calibration of GPS/levelling and Geoid, because in this case F=0 and the BQUNE method does not solve our problem. In order to avoid this problem, this method should be modified so that the F matrix exists. In order to modify this method the following technique suggested recently by Eshagh (2006) is introduced.

First of all, the transformation from the *W* to the residual vector should be created, namely,

$$\gamma_i = F_i QB^T C^{-1} \overline{W} = F_i \varepsilon, \quad (26)$$

All of the parameters have been introduced before. We mean we can separate the residual vector instead; like the one we did in Gauss-Markov model. In this case, the F_i is defined like the one used in the Gauss-Markov model Equation (8). Now the γ_i is normally distributed with zero mean and variance of

$$E\{\gamma_i \gamma_i^T\} = \sigma_i^2 F_i QB^T C^{-1} BQF_i, \quad (27)$$

we denote

$$P_i = F_i QB^T C^{-1} BQF_i, \quad (28)$$

In this method, F_i has the role of a separator, that separates the residuals of the corresponding group of observations, and produces a P_i for each group, which plays the role of variance covariance matrix corresponding to the group. The variance components can easily be computed by this method too (Equation 8). A similar argument can be put forward for the Gauss-Helmert model. It suffices to convert the model by using a projection to the condition model.

3 SPECIALIZATION OF PROBLEM TO GPS/LEVELLING AND GEOID

In the calibration of GPS/levelling and Geoid precision the first design matrix can be introduced in different ways. One may choose 4, 5 and 7-parameter models in order to remove the biases and title of the geoid. These models may have the following first design matrices

$$\boldsymbol{a}_i^T \boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{x}_1 \cos\varphi_i \cos\lambda_i + \boldsymbol{x}_2 \cos\varphi_i \sin\lambda_i + \boldsymbol{x}_3 \sin\varphi_i, \quad (29)$$

and rarely its five-parameter extension

$$\boldsymbol{a}^T{}_i \boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{x}_1 \cos\varphi_i \cos\lambda_i + \boldsymbol{x}_2 \cos\varphi_i \sin\lambda_i + \boldsymbol{x}_3 \sin\varphi_i + \boldsymbol{x}_4 \sin^2\varphi_i, \quad (30)$$

$$\boldsymbol{a}_i = \begin{pmatrix} \cos\varphi_i \cos\lambda_i \\ \cos\varphi_i \sin\lambda_i \\ \sin\varphi_i \\ \cos\varphi_i \sin\varphi_i \cos\lambda_i / K_i \\ \cos\varphi_i \sin\varphi_i \sin\lambda_i / K_i \\ \sin^2\varphi_i / K_i \\ 1 \end{pmatrix} \quad \text{and}$$

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}, \quad (31)$$

where φ and λ are the horizontal geodetic coordinates of the network or baseline points and

$$K_i = \left(1 - e^2 \sin^2 \varphi_i\right)^{1/2}, \quad (32)$$

where e is the first eccentricity of the reference ellipsoid. The second design matrix in this cases is defined as follows

$$B = \begin{bmatrix} I_n & -I_n & -I_n \end{bmatrix} \quad (33)$$

where, n is the number of GPS/levelling points. The stochastic model of the variance components can be written as

$$Q = \sigma_h \begin{bmatrix} Q_h & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \sigma_H \begin{bmatrix} 0 & 0 & 0 \\ 0 & Q_H & 0 \\ 0 & 0 & 0 \end{bmatrix} + \sigma_N \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Q_N \end{bmatrix} = \begin{bmatrix} \sigma_h Q_h & 0 & 0 \\ 0 & \sigma_H Q_H & 0 \\ 0 & 0 & \sigma_N Q_N \end{bmatrix} \quad (34)$$

therefore, we can write

$$BQB^T = \sigma_h Q_h + \sigma_H Q_{Hh} + \sigma_N Q_N = Q_W, \quad (35)$$

In this case, the Equation (2) will be converted to

$$X = (A^T Q_w^{-1} A)^{-1} A^T Q_w^{-1} W, \quad (36)$$

and the residual vector will be (Equation 3)

$$\varepsilon = QB^T Q_w^{-1} (I - A^0) W, \quad (37)$$

where

$$A^0 = AA^- = A(A^T Q_w^{-1} A)^{-1} A^T Q_w^{-1}, \quad (38)$$

The Equation (9) can be re-written as

$$K_0 = (I - A^0) Q_w (I - A^0)^T = Q_w (I - A^0)^T, \quad (39)$$

And its inverse becomes

$$K_0^{-1} = (Q_w (I - A^0)^T)^{-1} = Q_w^{-1}, \quad (40)$$

Finally the variance components can be computed by the BQUE method as Equation (7) and (8) will be transformed to

$$S_{ij} = \mathrm{trace}(Q_w^{-1} K_i Q_w^{-1} K_j), \quad (41)$$

$$q_i = \bar{W}^T Q_w^{-1} K_i Q_w^{-1} \bar{W}, \quad i=h, H, N \quad (42)$$

The variance components can easily be computed using Equation (6). For the Modified Best Quadratic Unbiased Non-negative Estimation we have

$$\gamma_i = F_i Q B^T Q_w^{-1} \bar{W} = F_i \varepsilon, \quad (43)$$

where F_i is defined as before. The Equation (28) can be reformulated as

$$P_i = F_i Q B^T Q_w^{-1} B Q F_i, \quad (44)$$

Having computed these parameters, the Equation (17) can be used for computing the variance components.

3-1 NEGATIVITY NUMBER

Negativity number of variance components estimation procedure is defined as the square roots of summation of square values of negative components. The negativity number is zero if all of the components come out positive. It ranges from zero to infinity.

$$N = \sqrt{\sum_{i=1}^{n} (\sigma_i)^2}, \text{ where } \sigma_i \langle 0 \quad (45)$$

The negativity number is used in our computations as a control criterion, so that whenever even one component comes out negative it gets a non-zero value and immediately the non-negative method of MBQUNE is used for the solution.

4 MATLAB FUNCTION ESTIMATING VC

For computing the variance components a MATLAB function has been programmed, so that its input argument is the name of the data file, the parametric model which the user prefers to use as the corrective surface, for example, one can choose, 4, 5, or 7-parameter corrective surfaces. The output of the function is the mean values of the ellipsoidal, orthometric, and geoidal height accuracies, and the number of iteration needed to converge the variance components. If negative values are obtained the program informs the user to obtain the negative values using the usual BQUE methods and presents the negativity number. Having obtained the negative values, the program is switched to the method of Modified Best Quadratic Unbiased Non-negative Estimation and the variance components will be solved positively. The function also presents the behaviour of the variance components during iterations. Figure 1 shows the flowchart of the VC program.

The data file includes 7 columns, where the first and second columns consist of the latitude and longitude of the GPS/levelling points, third, fourth and fifth columns are the ellipsoidal, orthometric and geoidal heights, respectively.

Sixth, seventh, and eighth columns are the diagonal values of the precision for the ellipsoidal, orthometric, and geoidal heights, respectively. In practice, fully-populated covariance matrices for each group of heights are usually not made available to the users or they are difficult to obtain. So, by ignoring the correlation between data, the diagonal matrix was used in modelling. However, the results in this case may be slightly optimistic compared the full CV matrix results (Fotopoulos, 2003).

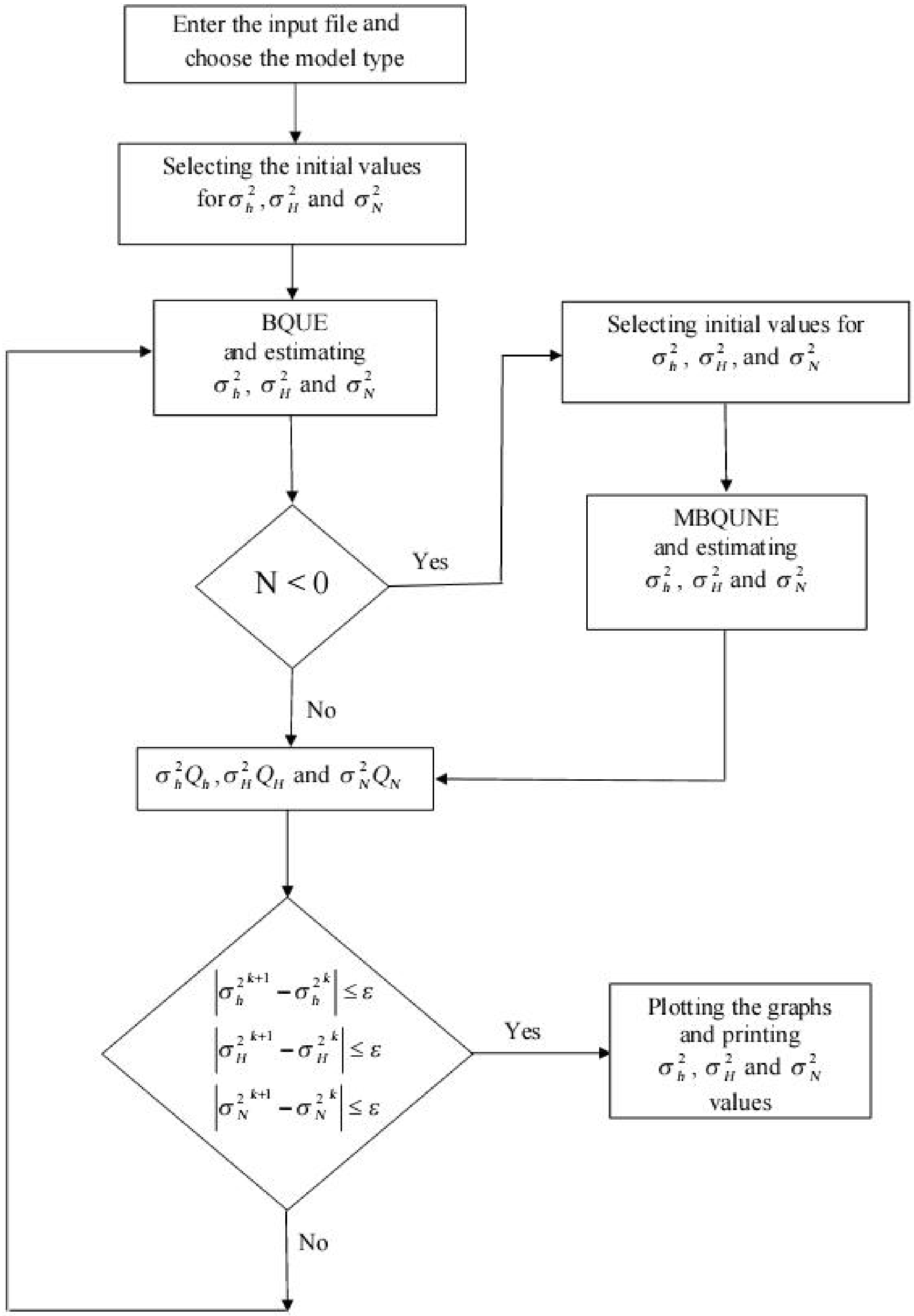

Figure 1a. The flowchart of the MTALAB variance component analysis program.

INPUT Data

Data file should have the following format:

phi lambda h H N dh dH dN

where,

Latitude/Longitude	**(in degrees)**
h= Ellipsoidal height	**(in meters)**
H= Orthometric height	**(in meters)**
N= Geoidal height	**(in meters)**
dh= Ellipsoidal height error	**(in meters)**
dH= Orthometric height error	**(in meters)**
dN= Geoidal height error	**(in meters)**

The name of the file including the above information is the first input argument
The number of parameters of the corrective surface to be used is the second argument
For example if the name of the data file is GPSLEVELING.txt and 5 parameters corrective surface model is used we should type

VC (´GPSLEVELING.txt´, 5)

OUTPUT

Sigma h=mean error of the ellipsoidal heights after VCE process (in meters)
Sigma H=mean error of the orthometric height after VCE process (in meters)
Sigma N=mean error of the geoidal height after VCE process (in meters)
and their plots.

Figure 1b. A sample input and output data which are used in VC software.

4-1 NUMERICAL INVESTIONS

The vertical test network consisting of 73 GPS/levelling benchmarks distributed throughout Iran was used for the VCE in this research. As seen in figure 2, distribution of vertical control is very mixed and too sparse in most parts.

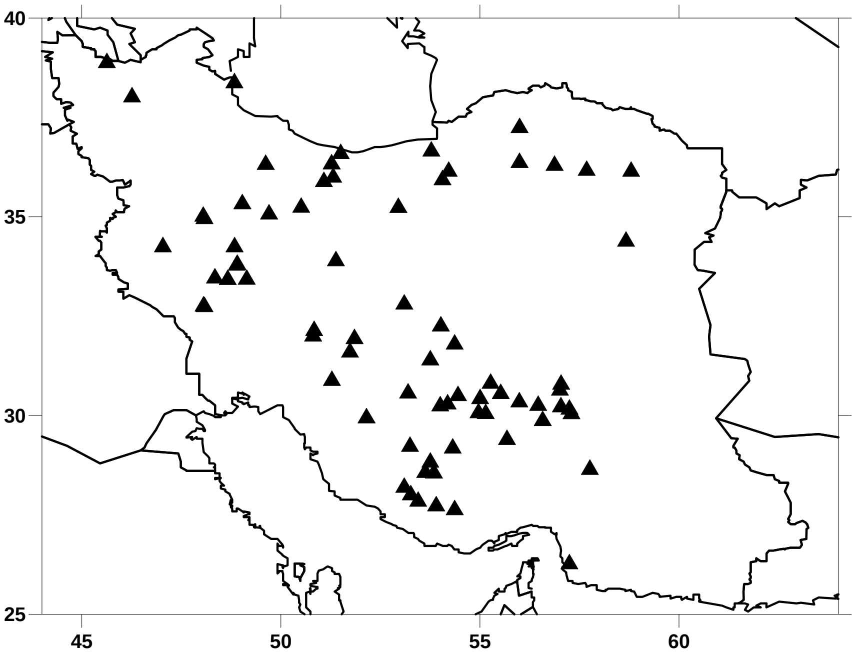

Figure 2. Location of the GPS/levelling points.

The diagonal weight matrix for the three-dimensional GPS coordinates was obtained from the recent output of the adjustment of the Iranian GPS network (Nankali, Personal communication). The overall average standard deviation is approximately 12.5 cm.

Also, we used the IRG04 precise gravimetric geoid model in this research. The IRG04 gravimetric geoid model was computed by using the least squares modification of Stokes formula based on the recent published GRACE based global geopotential model, the high-resolution Shuttle SRTM global digital terrain model, and a new Iranian gravity anomaly database (Kiamehr, 2006a and 2006b).

The a-priori error CV matrix, Q_N for the geoidal undulations was obtained through error propagation of an error grid. The variance values corresponding to the 73 stations of interest were obtained through bilinear interpolation of the error grid. The overall average standard deviation is approximately 9.3 cm.

The initial diagonal covariance matrix for the orthometric heights, Q_H comes directly from the rigorous national adjustment of the first and second order levelling measurements (Hamesh, 1991 and NCC, 2003). The overall average standard deviation is approximately 7 cm. Table 1 shows statistical analysis of height values in the study area.

Three different parametric models were selected for the investigations in this case study, namely the classic 4, 5 and 7 parameter models. Tables (2) and (3) and figure 3 show the result of variance components analysis and error estimation for three parameters (h, H and N) by using the various parametric models.

Table 1. Statistical analysis of the height values in the study area. Unit: m.

Parameter	Max.	Min.	Mean	SD	$\bar{\sigma}_i$
N	-25.695	18.799	-3.940	9.079	**0.093**
h	0.138	2517.352	1431.995	585.932	**0.125**
H	-0.9	2515.991	1428.022	588.614	**0.070**

Table 2. Estimated variance components using various parametric models. Unit: m.

Model type	Solution	Iteration	σ_h^2	σ_H^2	σ_N^2
4-parameter	Negative	13	3.72	0.36	1.97
5-parameter	Positive	18	4.24	0.72	1.06
7-parameter	Positive	20	4.59	0.93	0.57

Table 3. Estimated error for h, H and N using the various parametric models, Unit: meter.

Model type	σ_h^2	σ_H^2	σ_N^2
4-parameter	0.137	0.075	0.132
5-parameter	0.146	0.107	0.097
7-parameter	0.152	0.121	0.071

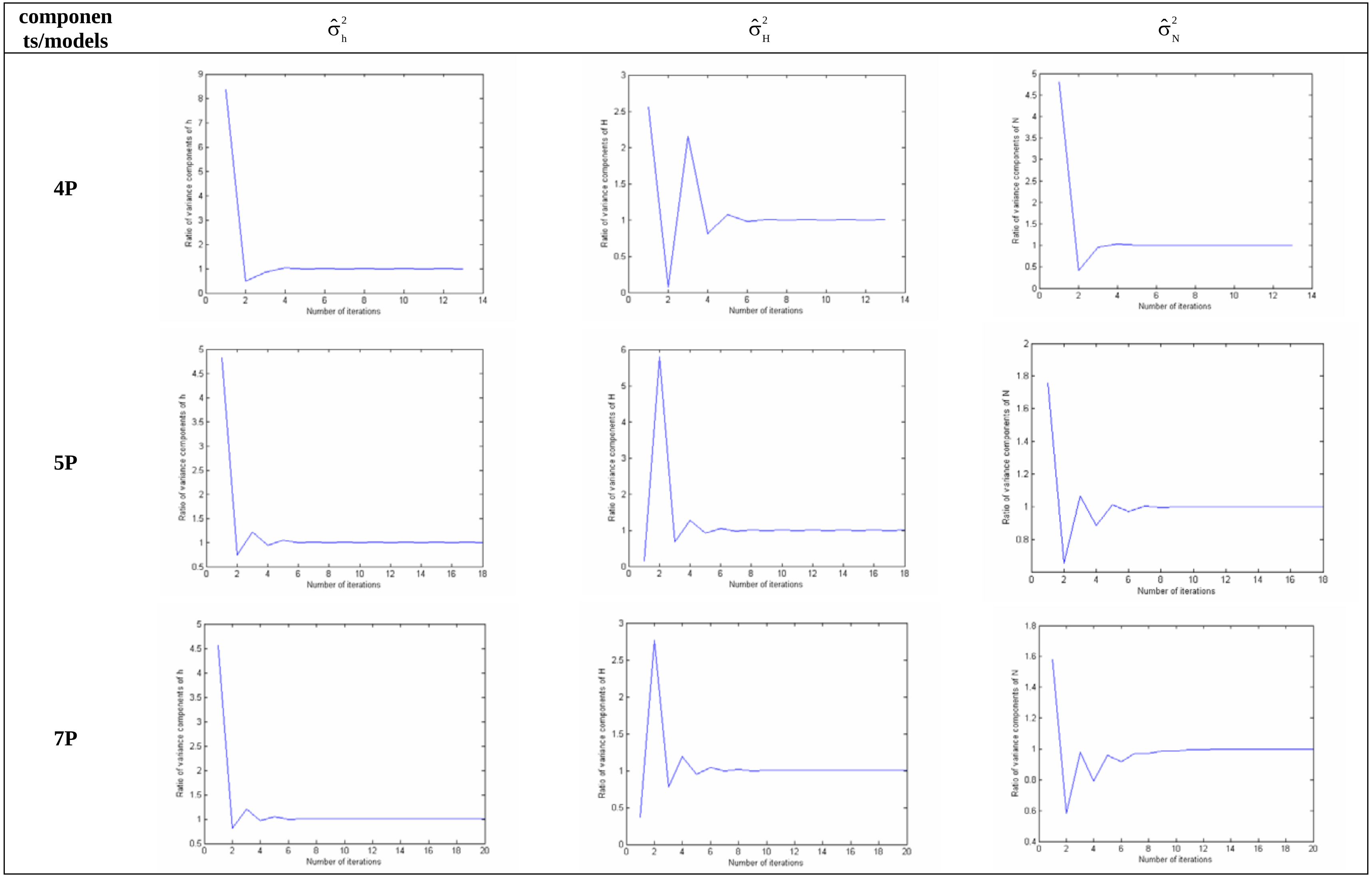

Figure 3. Iterations and estimated variance components using the different models. X and Y axes show the variance components and iteration numbers, respectively.

Table (2) shows the solution of the variance components for ellipsoidal, orthometric and geoidal heights. The results clearly show that the type of parametric model used in the combined least-squares adjustment affects the estimated values for the variance components. In both networks, the variance factor for the orthometric heights was influenced the most by the change in corrector surface, resulting in differences of several centimeters in the estimated variance factors from the use of a constant bias and the other parametric models.

The variance components come out negative when the 4-parameter model is used. In this case the variance components of the orthometric height come out negative and the rest are positive. This may indicate that an inadequate model was used for the systematic effects resulting in 'residual' biases that corrupt the performance of the VCE method. Another possibility is numerical instabilities caused by over-parameterization, which occurred when a complete fourth-order model was used in the test network. In any case, these first results are revealing as they suggest a means for identifying the inappropriateness of the tested corrector surface. The 4-parameter model gives negative VC so the MBQUNE method should be used for resolution of the problem and the solution comes to convergence after 13 iterations.

An estimated negativity number for the 4-parameter model is too small and about 0.076. The magnitude of the negativity number is related to the number and magnitude of negative variance components. According to the numerical studies if the negativity number comes to a very small value, the estimated variance components by MBQUNE are closer to the well-known BQUE method.

The 5 and 7-parameter models, give positive VC, and convergences with the 18 and 20 iterations were required, respectively. However, based on the result of combined adjustment (Kiamehr, 2006a), the 7-parameter model gave a better standard deviation for fitting, compared to the two models mentioned. So, the estimated VC from the 7-parameter solution can be used as final values for any practical investigation about accuracies of orthometric, geoidal and ellipsoidal heights in future. Also, by comparing the statistical analysis of the mean standard deviation values ($\overline{\sigma}_i$) in table (1) and their estimated VC in table (3), it is clear that the primary chosen accuracies for all height values are very optimistic (specially for the orthometric height) and could be considered in any future investigations.

Figure 3 shows the ratio of the variance components during iteration. This ratio changes by using the different model. As can be seen, for the ellipsoidal height parameter, the ratio of initial and the variance components computed at the first iteration is large for all models but they are different in magnitude. Larger oscillations can be seen in the ratio of VCs for orthometric heights and this variation becomes larger when the 4-parameter model has been used. As mentioned before, it may be because of choosing too pessimistic initial accuracies for orthometric heights. The VC ratio of the geoidal height oscillates is larger when the 5 and 7-parameters models are used compared to the 4-parameter model.

In general, the higher order parameter models need more iteration for convergence of the solution (See, Table 2). In these studies we have considered $\varepsilon = 0.00001$ as a criterion for convergence, so that if the difference between the two estimated last variance components becomes smaller than ε the solution is stopped and accepted.

Another interesting point in Figure 2 is the reverse behaviour of this ratio in orthometric height. The ratio increases at the first iteration when 5 or 7-parameter models are used while it decreases for the 4-parameter model; this behaviour depends on the type of estimators. However, the behaviour of the other VC ratios is more or less similar. Also, the VCs of the ellipsoidal and geoidal heights converge faster when MBQUNE is used.

5 CONCLUSIONS

A modified version of the Best Quadratic Unbiased Non-negative Estimator (MBQUNE) was successfully applied for estimation of the variance components through combined adjustment of the GPS/levelling and geoidal heights in Iran. A general MATLAB function is presented which can apply for estimation of the VC based on the three different corrective parametric models. The user should check different models to see the effect of using different corrective surfaces in the estimation of the VC and finding the best fitting model. Based on the numerical results by the MBQUNE approach, the absolute accuracy of the Iranian gravimetric geoid IRG04 is estimated to be about 27 cm through the combined adjustment of the 73 GPS/levelling and geoidal data. Also, the estimated accuracies for the geodetic and orthometric heights come up to 39 and 35 cm, based on the 7-parameter model, respectively.

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