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Asymptotic Behaviors of the Lorenz Curve for Left Truncated and Dependent Data

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Abstract

The purpose of this paper is to provide some asymptotic results for nonparametric estimator of the Lorenz curve and Lorenz process for the case in which data are assumed to be strong mixing subject to random left truncation. First, we show that nonparametric estimator of the Lorenz curve is uniformly strongly consistent for the associated Lorenz curve. Also, a strong Gaussian approximation for the associated Lorenz process is established under appropriate assumptions. Using this strong Gaussian approximation, a law of the iterated logarithm for the Lorenz process is also derived.

Keywords: Law of the iterated logarithm; Lorenz curve; Strong Gaussian approximation; Strong mixing; Truncated data

Introduction

Pietra [25] and Gastwirth [18] independently introduced the *Lorenz curve* corresponding to a non-negative random variable (rv) X with a distribution function (df) F, quantile function $Q(p)$ and finite mean $E(X)=\mu$ as:

$$L_F(t) := \frac{1}{\mu}\int_0^t Q(s)ds, \quad 0 \le t \le 1.$$

In econometrics, with X representing income, $L(t)$ gives the fraction of total income that the holders of the lowest t^{th} fraction of income possesses. Most of the measures of income inequality are derived from the Lorenz curve. An important example is the Gini index associated with F defined by

$$G_F := \frac{\int_0^1 [u - L_F(u)]du}{\int_0^1 u du} = 1 - 2(CL)_F,$$

where $(CL)_F = \int_0^1 L_F(u)du$ is the *cumulative Lorenz curve* corresponding to F. This is a ratio of the area between the Lorenz curve and the 45° line to the area under the 45° line. The numerator is usually called the *area of concentration*. Kendall and Stuart [21] showed that this is equivalent to a ratio of a measure of dispersion to the mean. In general, these notions are useful for measuring concentration and inequality in distributions of resources, and in size distributions. For a list of applications in different areas, we refer the

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readers to Csörgő and Zitikis [9].

To estimate the Lorenz curve, one can use the *Lorenz statistic* $L_n(y)$ defined by

$$L_n(y) = \frac{1}{\mu_n}\int_0^y Q_n(u)du, \quad 0 \le y \le 1,$$

where μ_n is the sample mean and $Q_n(y)$ is the empirical quantile function constructed from independent and identically distributed (i.i.d.) sample taken from F.

Goldie [19] proved the uniform consistency of L_n to L_F and derived the weak convergence of the *Lorenz process* $l_n(t) := \sqrt{n}\left[L_n(t) - L_F(t)\right], 0 \le t \le 1$ to a Gaussian process under suitable conditions. Csörgő et al. [6] gave a unified treatment of strong and weak approximations of the Lorenz and other related processes. In particular, they established a strong invariance principle for the Lorenz process, by which Rao and Zhao [26] derived one of their two versions of the law of the iterated logarithm (LIL) for the Lorenz process. Different versions of the LIL under weaker assumptions are also obtained by Csörgő and Zitikis ([9], [11]). In Csörgő and Zitikis [10], confidence bands for the Lorenz curve that are based on weighted approximations of the Lorenz process are constructed. Csörgő et al. [7], obtained weak approximations for Lorenz curves under random right censorship. Strong Gaussian approximations for the Lorenz process when data are subject to random right censorship and left truncation was established by Tse [27], he is also derived a functional LIL for the Lorenz process.

However, in most economic situations, the basic sequence of observations may not be independent. It is more realistic to assume some form of dependence among the data are observed. Csörgő and Yu [8], obtained weak approximations for Lorenz curves and its inverse under the assumption of mixing dependence. Glivenko-Cantelli-type asymptotic behavior of the empirical generalized Lorenz curves based on random variables forming a stationary ergodic sequence with deterministic noise were considered by Davydov and Zitikis[12]. Davydov and Zitikis [13] established a large sample asymptotic theory for the empirical generalized Lorenz curves when observations are stationary and either short-range or long-range dependent. Strong laws for the generalized absolute Lorenz curves when data are stationary and ergodic sequences established by Helmers and Zitikis [20]. Based on the generalized Lorenz curves Davydov et al. [14] proposed a statistical index for measuring the fluctuations of a stochastic process. They developed some of the asymptotic theory of the statistical index in the case where the stochastic process is a Gaussian process with stationary increments and a nicely behaved correlation function. The uniform strong convergence rate of the Lorenz curve estimator under strong mixing hypothesis is obtained by Fakoor et al. [17]. They also established a strong Gaussian approximation for the Lorenz process, by which they derived a functional LIL for the Lorenz process, under the assumption of strong mixing. The counterpart of these results for the censored dependent model was established by Bolbolian et al. [2].

The purpose of this paper is to provide some asymptotic results for Lorenz process $l_n(t)$ for the case in which data are assumed to be strong mixing subject to random left truncation.

Consider a sequence of rv's $X_1, X_2, ..., X_n$ with common unknown absolutely continuous df F and finite mean μ. These rv's are regarded as the lifetimes of the items under study which may not be mutually independent. Among the different forms in which incomplete data appear, right censoring and left truncation are two common ones. Left truncation may occur if the time origin of the lifetime precedes the time origin of the study. Only subjects that fail after the start of the study are being observed, otherwise they are left truncated. This means that some subjects are sampled, while others are neglected. This model arises in various fields, e.g., astronomy, economy and medical studies (see, e.g. [28]). Let $T_1, T_2, ..., T_n$ be a sequence of i.i.d. random variables with continuous df G, they are also assumed to be independent of the rv's X_i's. In the left truncation model, (X_i, T_i) is observed only when $X_i \ge T_i$. Let $(X_1, T_1), ..., (X_n, T_n)$ be a sample which one observes (i.e., $X_i \ge T_i$), and $\gamma := \mathrm{P}(T_1 \le X_1) > 0$, where P is the absolute probability (related to the N -sample). Note that n itself is a rv and that γ can be estimated by $\frac{n}{N}$ (although this estimator cannot be calculated since N is unknown). Assume, without loss of generality, that X_i and T_i are nonnegative random variables, $i = 1, ..., N$. For any df L denotes the left and right endpoints of its support by $a_L = \inf\{x : L(x) > 0\}$ and $b_L = \sup\{x : L(x) < 1\}$, respectively. Then under the current model, as discussed by Woodroofe [28], we assume that $a_G \le a_F$ and $b_G \le b_F$. Define

$$\text{(1.1)} \quad \begin{aligned} C(x) &= P(T_1 \le x \le X_1 | T_1 \le X_1) \\ &= P(T_1 \le x \le X_1) = \gamma^{-1} G(x)(1 - F(x)), \end{aligned}$$

where $P(.) = \mathrm{P}(.|n)$ is the conditional probability (related to the n-sample) and consider its empirical estimate

$$\text{(1.2)}\quad C_n(x)=n^{-1}\sum_{i=1}^{n} I(T_i \le x \le X_i),$$

where I(.) is the indicator function. Then the product-limit (PL) estimator $\hat{F}_n$ of F is given by

$$\text{(1.3)}\quad \hat{F}_n(x)=1-\prod_{X_i \le x}\left(1-\frac{1}{nC_n(X_i)}\right).$$

The cumulative hazard function $\Lambda(x)$ is defined by

$$\text{(1.4)}\quad \Lambda(x) = \int_0^x \frac{dF(u)}{1-F(u)}.$$

Let

$$\text{(1.5)}\quad \begin{aligned} F^*(x) &= \mathrm{P}(X_1 \le x \mid T_1 \le X_1) \\ &= P(X_1 \le x) = \gamma^{-1}\int_0^x G(u)dF(u), \end{aligned}$$

be the df of the observed lifetimes. Its empirical estimator is given by

$$F_n^*(x) = n^{-1}\sum_{i=1}^{n} I(X_i \le x).$$

On the other hand, the df of the observed T_i's is given by

$$\begin{aligned} G^*(x) &= \mathrm{P}(T_1 \le x \mid T_1 \le X_1) \\ &= P(T_1 \le x) = \gamma^{-1}\int_0^\infty G(x \wedge u)dF(u), \end{aligned}$$

and is estimated by

$$G_n^*(x) = n^{-1}\sum_{i=1}^{n} I(T_i \le x).$$

It then follows from (1.1) and (1.2) that

$$\text{(1.6)}\quad \begin{aligned} C(x) &= G^*(x) - F^*(x), \\ C_n(x) &= G_n^*(x) - F_n^*(x-). \end{aligned}$$

Finally (1.1), (1.4) and (1.5) give

$$\Lambda(x) = \int_0^x \frac{dF^*(u)}{C(u)}.$$

Hence, a natural estimator of Λ is given by

$$\hat{\Lambda}_n(x) = \int_0^x \frac{dF_n^*(u)}{C_n(u)} = \sum_{i=1}^{n}\frac{I(X_i \le x)}{nC_n(X_i)},$$

which is the usual so-called Nelson-Aalen estimator of Λ. Moreover, $\hat{\Lambda}_n$ is the cumulative hazard function of the PL estimator $\hat{F}_n$ defined in (1.3).

The quantile function Q and its empirical counterpart Q_n are defined by

$$\text{(1.7)}\quad \begin{aligned} &Q(p)=\inf\{x \in \Re;\ F(x) \ge p\} \quad \text{and} \\ &Q_n(p)=\inf\{x \in \Re;\ \hat{F}_n(x) \ge p\} \end{aligned}$$

Suppose that $0 < p_0 \le p_1 < 1$. We defined the Lorenz curve corresponding to rv X as:

$$L_F(p) := \frac{1}{\mu}\int_{p_0}^{p} Q(s)ds, \quad p_0 \le p \le p_1,$$

where $\mu = \int_{p_0}^{p_1} Q(s)ds$. Therefore the natural estimator for the Lorenz curve $L_F(p)$ is

$$L_n(p) := \frac{1}{\mu_n}\int_{p_0}^{p} Q_n(s)ds, \quad p_0 \le p \le p_1,$$

where $\mu_n = \int_{p_0}^{p_1} Q_n(s)ds$.

The main aims of this paper are to derive strong uniform consistency of the Lorenz statistic and strong Gaussian approximation for Lorenz process, for the case in which data are assumed to be dependent subject to random left truncation. As a result of our strong Gaussian approximation, we obtain a functional LIL for the Lorenz process.

In this paper we consider the strong mixing dependence, which amounts to a form of asymptotic independence between the past and the future as shown by its definition.

Definition 1. Let $\{X_i, i \ge 1\}$ denote a sequence of random variables. Given a positive integer m, set

$$\text{(1.8)}\quad \begin{aligned} &\alpha(m) = \\ &\sup_{k \ge 1}\left\{|P(A \cap B) - P(A)P(B)|;\ A \in \Im_1^k,\ B \in \Im_{k+m}^\infty\right\} \end{aligned}$$

where $\Im_i^k$ denote the σ-field of events generated by $\{X_j;\ i \le j \le k\}$. The sequence is said to be strong mixing (α-mixing) if the mixing coefficient $\alpha(m) \to 0$ as $m \to \infty$.

Among various mixing conditions used in the literature, strong mixing is reasonably weak and has many practical applications (see, e.g. [16], [4] or [5] for more details). In particular, Masry and Tjostheim [24] proved that, both ARCH processes and nonlinear additive AR models with exogenous variables, which are particularly popular in finance and econometrics, are stationary and strong mixing.

Now we introduce our main assumption that is used to state our results gathered below for easy reference.

A. $\{X_i, i \ge 1\}$ is a sequence of stationary strong mixing rv's with mixing coefficient $\alpha(n) = O(e^{-(\log n)^{1+\nu}})$ for some $\nu > 0$.

In the next Section, we present our main results.

Results

In this section we first derive strong uniform consistency of the Lorenz statistic and strong Gaussian approximation for Lorenz process, for the case in which data are assumed to be dependent subject to random left truncation and finally as a result of our strong Gaussian approximation, we obtain a functional LIL for the Lorenz process.

Theorem 1 below proves the uniform strong consistency with rate of the estimator L_n.

Theorem 1. Let $0 < p_0 \le p_1 < 1$. Under Assumption **A**, assuming that $F' = f$ is bounded away from zero on $[Q(p_0)-\delta, Q(p_1)+\delta]$, for some $\delta > 0$. Then

$$(2.1)\quad \sup_{p_0 \le p \le p_1} \left| L_n(p) - L_F(p) \right| = O\left(\sqrt{\frac{\log \log n}{n}} \right) \ a.s.$$

Proof. An elementary computation shows that,

$$(2.2)\quad \begin{aligned} L_n(p) - L_F(p) &= \frac{1}{\mu_n} \int_{p_0}^{p} \left[Q_n(s) - Q(s) \right] ds \\ &\quad - \frac{\mu_n - \mu}{\mu_n} L_F(p). \end{aligned}$$

It is easy to see that,

$$(2.3)\quad \mu_n - \mu = \int_{p_0}^{p_1} \left[Q_n(s) - Q(s) \right] ds.$$

Now, by using (2.2), (2.3) and Lemma 3 of [23], we obtain the results. □

For construct strong Gaussian approximation we first introduce the following Gaussian process, which plays an important role to present our strong approximation.

Let $g_j(s) = I(X_j \le s) - F^*(s), j \ge 0,$

$$(2.4)\quad \begin{aligned} \Gamma(s,s') &= \mathrm{cov}(g_1(s), g_1(s')) \\ &\quad + \sum_{j=2}^{\infty} \left[\mathrm{cov}(g_1(s), g_j(s')) + \mathrm{cov}(g_1(s'), g_j(s)) \right]. \end{aligned}$$

Define, for $0 \le t \le b$, two parameter mean zero Gaussian process

$$(2.5)\quad B(t,n) := \frac{K(t,n)/\sqrt{n}}{C(t)} + \int_0^t \frac{K(u,n)/\sqrt{n}}{C^2(u)} dC(u),$$

where $\{K(s,t), 0 \le s,t \le b\}$ is a Kiefer process in Theorem 3 of [15] with covariance function

$$\Gamma^*(t,t',s,s') = \min(t,t') \Gamma(s,s'),$$

and $\Gamma(s,s')$ given by (2.4).

We now restate below a strong approximation by Bolbolian et al. [3] for the normed PL-quantile process $\rho_n(u) := \sqrt{n} f(Q(u))[Q(u) - Q_n(u)]$ by a two parameter Gaussian process at the rate $O((\log n)^{-\lambda})$, for some $\lambda > 0$. The statements are conditional on the observed sample size *n*.

Theorem 2. (Bolbolian et al. [3]) Let $0 < p_0 \le p_1 < 1$. Under Assumption **A**, assume that F is Lipschtiz continuous and that F is twice continuously differentiable on $[Q(p_0)-\delta, Q(p_1)+\delta]$, for some $\delta > 0$, such that f is bounded away from zero, then there exists a two parameter mean zero Gaussian process $B(t,u)$ for $t,u \ge 0$, such that,

$$\sup_{p_0 \le p \le p_1} \left| \rho_n(p) - (1-p) B(Q(p), n) \right| = O\left((\log n)^{-\lambda} \right) \text{ a.s.,}$$

for some $\lambda > 0$ □

We will give strong Gaussian approximation of the Lorenz process over restricted interval $[p_0, p_1]$ for fixed $0 < p_0 \le p_1 < 1$.

In the full model, Langberg et al. [22] define the *total time on test transform* curve corresponding to a continuous distribution F on $[0,\infty), H_F^{-1}(p)$, for $p \in [0,1]$ as

$$\begin{aligned} H_F^{-1}(p) &= \int_0^p (1-y) dQ(y) \\ &= (1-p)Q(p) + \int_0^p Q(y) dy, \quad Q(0) = 0. \end{aligned}$$

Obviously, $H_F^{-1}(p) \le H_F^{-1}(1) := \lim_{p \uparrow 1} H_F^{-1}(p) = \mu$. For the our model, we modify the definition of $H_F^{-1}(p)$ as

$$(2.6)\quad \begin{aligned} H_F^{-1}(p) &= (p_1 - p) Q(p) \\ &\quad + \int_{p_0}^{p} Q(y) dy, \quad p \in [p_0, p_1]. \end{aligned}$$

As $p_0 \downarrow 0$ and $p_1 \uparrow 1$, $H_F^{-1}(p_1) \to \int_0^1 Q(y)dy = \mu$. We can regard $H_F^{-1}(p_1)$ as a surrogate for the finite mean μ. A natural estimator for $H_F^{-1}(p)$ is

$$H_n^{-1}(p) = (p_1 - p)Q_n(p) + \int_{p_0}^{p} Q(y)dy, \quad p \in [p_0, p_1].$$

In the next theorem, we construct a two parameter mean zero Gaussian process that strongly uniformly approximate the empirical process $1_n(p)$.

Theorem 3. Let $0 < p_0 \le p_1 < 1$. Under Assumption **A**, assume that F is Lipschtiz continuous and that F is twice continuously differentiable on $[Q(p_0)-\delta, Q(p_1)+\delta]$, for some $\delta > 0$, such that f is bounded away from zero. Then there exists a two parameter mean zero Gaussian process $B(t,u)$ for $t,u \ge 0$, such that, almost surely,

$$\sup_{p_0 \le p \le p_1} \left| 1_n(p) - \frac{1}{H_F^{-1}(p_1)} \left(\int_{p_0}^{p} \frac{(p_1 - y)B(Q(y),n)}{f(Q(y))} dy - L_F(p) \int_{p_0}^{p_1} \frac{(p_1 - y)B(Q(y),n)}{f(Q(y))} dy \right) \right| = O\left((\log n)^{-\lambda} \right), \tag{2.7}$$

for some $\lambda > 0$.

Proof. See the Appendix. □

The next theorem gives a functional LIL for the Lorenz process. We work on the probability space of Theorem 3. Let $D[a,b]$ $D[a,b]$ be the space of functions on $[a,b]$ that are right continuous and have left limits and B is the unit ball in the reproduce kernel Hilbert space $H(\Gamma^*)$.

Theorem 4. Suppose that conditions of Theorem 3 are satisfied. On a rich enough probability space, $l_n(.)/\sqrt{2\log\log n}$ is almost surly relatively compact in $D[p_0,p_1]$ with respect to the supremum norm and its set of limit points is

$$G = \left\{ g_h : g_h(u) = \frac{1}{H_F^{-1}(p_1)} \int_{p_0}^{u} \frac{h(y)}{f(Q(y))} dy - L_F(u) \int_{p_0}^{p_1} \frac{h(y)}{f(Q(y))} dy, p_0 \le u \le p_1, h \in H \right\},$$

where

$$H = \left\{ h : [p_0, p_1] \to \Re, h(u) = \frac{g(u)}{C(u)} + \int_0^u \frac{g(x)}{C^2(x)} dC(x) : g \in B \right\}.$$

Proof. Theorem 4 follows at once from (2.7) and Theorem A in [1]. □

Appendix

In establishing Theorem 3, we were aided by some ideas found in [27], but first we start with the following lemmas which is necessary for achieving the establishment of the our results.

Lemma 1. Suppose the conditions of Theorem 2 are satisfied. We have,

$$\lim_{n\to\infty} \sup_{p_0 \le p \le p_1} \left| H_n^{-1}(p) - H_F^{-1}(p) \right| = O\left(\sqrt{\frac{\log\log n}{n}} \right) \text{ a.s.}$$

Proof. By Lemma 3 of [25], we have,

$$\sup_{p_0 \le p \le p_1} \left| H_n^{-1}(p) - H_F^{-1}(p) \right| \le \sup_{p_0 \le p \le p_1} \left[(p_1 - p)|Q_n(p) - Q(p)| \right] + \sup_{p_0 \le p \le p_1} \int_{p_0}^{p} |Q_n(y) - Q(y)| dy = O\left(\sqrt{\frac{\log\log n}{n}} \right) \text{ a.s. } □$$

Next, define the normed total time on test empirical process $t_n(p)$ by

$$t_n(p) = \sqrt{n}\left[H_n^{-1}(p) - H_F^{-1}(p) \right], \quad p \in [p_0, p_1].$$

Lemma 2 characterize the asymptotic limit of $t_n(p)$.

Lemma 2. Suppose the conditions of Theorem 2 are satisfied. Then there exists a two parameter mean zero Gaussian process $B(t,u)$ for $t,u \ge 0$, such that,

$$\sup_{p_0 \le p \le p_1} \left| t_n(p) - \left(\int_{p_0}^{p} \frac{(p_1 - y)B(Q(y), n)}{f(Q(y))} dy \right. \right.$$

$$+\frac{(p_1\text{-}p)^2 B(Q(p),n)}{f(Q(p))}\Bigg)\Bigg| = O\left((\log n)^{-\lambda}\right), \quad \text{a.s.}$$

Proof. Proof of this lemma can be done using similar augment of Lemma 3.2 in [27], we therefore omit the proof. □

Next, we define the scaled total time on test transform, its statistic and associated empirical process corresponding to F.

$$(3.1)\quad W_F(p):=\frac{H_F^{-1}(p)}{H_F^{-1}(p_1)},\quad W_n(p):=\frac{H_n^{-1}(p)}{H_n^{-1}(p_1)},$$

and

$$w_n(p):=\sqrt{n}\left[W_n(p)\text{-}W_F(p)\right]$$

for $p\in[p_0,p_1]$.

The following lemmas give the strong uniform consistency of $W_n(p)$ and strong Gaussian approximation of the scaled total time on test empirical process respectively.

Lemma 3. Suppose that conditions of Theorem 2 are satisfied. We have,

$$\sup_{p_0\le p\le p_1}\left|W_n(p)\text{-}W_F(p)\right| = O\left(\sqrt{\frac{\log\log n}{n}}\right) \quad \text{a.s.}$$

Proof. By triangular inequality and Lemma 1, the left hand side is bounded by

$$\sup_{p_0\le p\le p_1}\left|\frac{H_n^{-1}(p)}{H_n^{-1}(p_1)}-\frac{H_n^{-1}(p)}{H_F^{-1}(p_1)}\right|$$

$$+\sup_{p_0\le p\le p_1}\left|\frac{H_n^{-1}(p)}{H_F^{-1}(p_1)}-\frac{H_F^{-1}(p)}{H_F^{-1}(p_1)}\right|$$

$$\le \sup_{p_0\le p\le p_1}\left|H_n^{-1}(p)\frac{H_F^{-1}(p_1)-H_n^{-1}(p_1)}{H_n^{-1}(p_1)H_F^{-1}(p_1)}\right|$$

$$+\sup_{p_0\le p\le p_1}\left|\frac{1}{H_F^{-1}(p_1)}\left[H_F^{-1}(p)-H_n^{-1}(p)\right]\right|$$

$$=O\left(\sqrt{\frac{\log\log n}{n}}\right) \quad \text{a.s.} \quad □$$

Lemma 4. Suppose that conditions of Theorem 2 are satisfied. Then there exists a two parameter mean zero Gaussian process $B(t,u)$ for $t,u\ge 0$, such that,

$$\sup_{p_0\le p\le p_1}\Bigg| w_n(p)\text{-}\frac{1}{H_F^{-1}(p_1)}\left(\int_{p_0}^{p}\frac{(p_1\text{-}y)B(Q(y),n)}{f(Q(y))}dy+\frac{(p_1\text{-}p)^2B(Q(p),n)}{f(Q(p))}\right)$$

$$+\frac{H_F^{-1}(p)}{\left(H_F^{-1}(p_1)\right)^2}\int_{p_0}^{p_1}\frac{(p_1\text{-}y)B(Q(y),n)}{f(Q(y))}dy\Bigg|$$

$$=O\left((\log n)^{-\lambda}\right) \text{ a.s.,}$$

for some $\lambda>0$.

Proof. Proof can be done along the lines of Lemma 3.5 of [27], we therefore omit the proof. □

Proof of Theorem 3. By Definition of the Lorenz curve corresponding to F in the our model and by using (2.6) and (3.1) we have

$$(3.2)\quad W_F(y)=\frac{(p_1\text{-}y)Q(y)}{\int_{p_0}^{p_1}Q(u)du}+L_F(y).$$

We have also

$$(3.3)\quad W_n(y)=\frac{(p_1\text{-}y)Q_n(y)}{\int_{p_0}^{p_1}Q_n(u)du}+L_n(y),\quad y\in[p_0,p_1].$$

Substituting (3.2) and (3.3) in Lemma 4, we obtain the result. □

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