Journal of Sciences, Islamic Republic of Iran 22(3): 257-266 (2011) http://jsciences.ut.ac.ir
University of Tehran, ISSN 1016-1104

A Strong Regular Relation on Γ-Semihyperrings

S.O. Dehkordi and B. Davvaz*

Department of Pure Mathematics, Faculty of Mathematics, Yazd University, Yazd , Islamic Republic of Iran

Received: 18 October 2010 / Revised: 24 August 2011 / Accepted: 12 October 2011

Abstract

The concept of algebraic hyperstructures introduced by Marty as a generalization of ordinary algebraic structures. In an ordinary algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. The concept of Γ-semihyperrings is a generalization of semirings, a generalization of semihyper-rings and a generalization of Γ-semirings. In this paper, we introduce an equivalence relation γ^* on a Γ-semihyperrings R and we show that it is strongly regular. Furthermore, R/γ^*, the set of all equivalence classes of this relation is a Γ/β^*-semiring. The relation γ^* is called the fundamental relation and the Γ-semiring R/γ^* is called the fundamental semiring. Fundamental relations are the main tools in the study of Γ-semihyperrings. We present some results about fundamental relations and fundamental semirings. Finally, we show that there is a covariant functor between the category of Γ-semihyperrings and the category of semirings.

Keywords: Γ-semihyperring; Γ-hyperring; Strongly regular relation; Fundamental semiring

Introduction

Algebraic hyperstructures represent a natural extension of classical algebraic structures. In 1934, at the 8th Congress of Scandinavian Mathematicians, Marty [14] has introduced, for the first time, the notion of *hypergroup*, using it in different contexts: algebraic functions, rational fractions and non-commutative groups. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. One of the first books, dedicated especially to hypergroups, is "Prolegomena of Hypergroup Theory", written by P. Corsini in 1993 [4]. Another book on "Hyperstructures and Their Representations", by T. Vougiouklis, was published one year later [18]. On the other hand, algebraic hyperstructure theory has a multiplicity of applications to other disciplines: geometry, graphs and hypergraphs, binary relations, lattices, groups, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, C-algebras, artificial intelligence, probabilities and so on. A recent book on these topics is "Applications of Hyperstructure Theory", by P. Corsini and V. Leoreanu, published by Kluwer Academic Publishers in 2003 [5]. Another book [7] is devoted especially to the study of hyperring theory. Several kinds of hyperrings are introduced and analyzed. The volume ends with an outline of applications in chemistry and physics, analyzing several special kinds

* Corresponding author, Tel.: +98(351)812711, Fax: +98(351)8210695, E-mail: davvaz@yazduni.ac.ir

of hyperstructures: e-hyperstructures and transposition hypergroups. Finally, we mention here another important book for the applications in Geometry and for the clearness of the exposition, written by W. Prenowitz and J. Jantosciak [17].

Let H be a non-empty set and $\circ : H \times H \to P^*(H)$ be a *hyperoperation*, where $P^*(H)$ is the family of all non-empty subsets of H. The couple $(H,\circ)$ is called a *hypergroupoid*. For any two non-empty subsets A and B of H and $x \in H$, we define $A \circ B = \bigcup_{a\in A, b\in B} a \circ b$, $A \circ \{x\} = A \circ x$ and $\{x\} \circ A = x \circ A$. A hypergroupoid $(H,\circ)$ is called a *semihypergroup* if for all a,b,c in H we have, $(a \circ b) \circ c = a \circ (b \circ c)$.

In addition, if for every $a \in H$, $a \circ H = H = H \circ a$, then $(H,\circ)$ is called a *hypergroup*. A non-empty subset K of a semihypergroup $(H,\circ)$ is called a *sub-semihypergroup* if it is a semihypergroup. In other words, a non-empty subset K of a semihypergroup $(H,\circ)$ is a sub-semihypergroup if $K \circ K \subseteq K$. We say that a hypergroup $(H,\circ)$ is *canonical* if

(i) it is commutative,

(ii) it has a scalar identity (also called scalar unit), which means that

$$\exists e \in H, \ \forall x \in H, \quad x \circ e = e \circ x = x,$$

(iii) every element has a unique inverse, which means that for all $x \in H$, there exists a unique $x^{-1} \in H$, such that $e \in x^{-1} \circ x \cap x \circ x^{-1}$,

(iv) it is reversible, which means that if $x \in y \circ z$, then there exist the inverses y^{-1} of y and x^{-1} of z, such that $z \in y^{-1} \circ x$ and $y \in x \circ z$.

A *Krasner hyperring* is an algebraic structure $(R,+,.)$ which satisfies the following axioms:

(i) $(R,+)$ is a canonical hypergroup,

(ii) $(R,.)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $x \circ 0 = 0 \circ x = 0$.

(iii) The multiplication is distributive with respect to the hyperoperation $+$.

In [2, 12], Davvaz et. al. studied the notion of a $\Gamma-$semihypergroup as a generalization of a semihyper-group. Many classical notions of semigroups and semihypergroups have been extended to $\Gamma-$semihyper-groups and a lot of results on $\Gamma-$semihypergroups are obtained.

The fundamental relation β^* was introduced on hypergroups by Koskas [13] for the first time and studied by many authors, for example see [4, 8, 9, 10 and 20]. The fundamental relation β^* is defined on hypergroups as the smallest equivalence relation so that the quotient would be a group.

Let H be a hypergroup and U be the set of all finite products of elements of H and define the relation β on H as follows:

$$x \,\beta\, y \text{ if and only if } \{x, y\} \subseteq u \text{ for some } u \in U.$$

Freni proved in [9] that for hypergroups we have $\beta^* = \beta$.

Vougiouklis in [18] defined the fundamental relation γ on hyperring R as the smallest equivalence relation on R such that the quotient R/γ^* is a fundamental ring. Let $(R,+,.)$ be a hyperring. Vougiouklis defined the relation as follows:

$$a \,\gamma\, b \text{ if and only if } \{a,b\} \subseteq u,$$

where u is a finite sum of finite products elements of R (u may be a sum of only one element), and proved that γ^* is the transitive closure of γ.

The fundamental equivalence relation extended to some classes of hyperrings by Spartalis and Vougiouklis [16]. In [10], Freni introduced a new strongly regular equivalence and a new characterization of the derived hypergroup of a hypergroup is determined.

By using a certain type of equivalence relations, we can connect $\Gamma-$semihyperrings to $\Gamma-$semirings. These equivalence relations are called *strong regular* relations. More exactly, starting with a $\Gamma-$semihyperring and using a *strong regular* relation, we can construct a $\Gamma-$semiring structure on the quotient set. Let R be a $\Gamma-$semihyperring and ρ be an equivalence relation on R. If R_1 and R_2 are non-empty subsets of R, then $R_1 \overline{\rho} R_2$ means that for every $x \in R_1$ there exists $y \in R_2$ such that $x \rho y$ and for every $y' \in R_2$ there exists $x' \in R_1$ such that $x' \rho y'$. $R_1 \overline{\overline{\rho}} R_2$ means that for every $x \in R_1$ and $y \in R_2$, we have $x \rho y$. A relation ρ on R is called right (resp. left) strongly regular if and only if $x \rho y$ implies that $(x+a)\overline{\overline{\rho}}(y+a)$ and $(x\alpha a)\overline{\overline{\rho}}(y\alpha a)$ for every $\alpha \in \Gamma$ and $a \in R$ (resp. $(a\alpha x)\overline{\overline{\rho}}(a\alpha y)$ and $(a+x)\overline{\overline{\rho}}(a+y)$), and R is called *strongly regular* if it is both left and right strongly regular.

By using a certain type of equivalence relations, we can connect Γ-semihyperrings to Γ-semirings. These equivalence relations are called *strong regular* relations.

More exactly, starting with a Γ-semihyperring and using a *strong regular* relation, we can construct a $\Gamma-$ semiring structure on the quotient set. Let R be a $\Gamma-$ semihyperring and ρ be an equivalence relation on R. If R_1 and R_2 are non-empty subsets of R, then $R_1 \overline{\rho} R_2$ means that for every $x \in R_1$ there exists $y \in R_2$ such that $x \rho y$ and for every $y' \in R_2$ there exists $x' \in R_1$ such that $x' \rho y'$. $R_1 \overset{=}{\rho} R_2$ means that for every $x \in R_1$ and $y \in R_2$, we have $x \rho y$. A relation ρ on R is called right (resp. left) strongly regular if and only if $x \rho y$ implies that $(x+a) \overset{=}{\rho} (y+a)$ and $(x \alpha a) \overset{=}{\rho} (y \alpha a)$ for every $\alpha \in \Gamma$ and $a \in R$ (resp. $(a \alpha x) \overset{=}{\rho} (a \alpha y)$ and $(a+x) \overset{=}{\rho} (a+y)$), and R is called *strongly regular* if it is both left and right strongly regular.

In this study, we introduce a relation γ on a given $\Gamma-$semihyperring R. and we show that the transitive closure of this relation is strongly regular, and the quotient R/γ^* is a Γ/β^*-semiring.

Let A and B be two non-empty subsets of $\Gamma-$ semihyperring R. We define

(i) $A+B=\{t \in R \mid x \in a+b \ a \in A, b \in B\}$,

(ii) $A\Gamma B=\{t \in R \mid t \in a\alpha b \ a \in A, b \in B, \alpha \in \Gamma\}$,

(iii) $A\Gamma^{\Sigma} B=\{t \in R \mid t \in \sum_{i=1}^{n} a_i \alpha_i b_i,$

$a_i \in A, b_i \in B, \alpha_i \in \Gamma \ n \in \mathbb{N}\}$.

Results

Definition 2.1. Let $(R,+)$ and $(\Gamma,+)$ be semihypergroups. Then R is said to be a $\Gamma-$semihyperring if there exists a mapping $R \times \Gamma \times R \to P^*(R)$ (image to be denoted by $a\alpha b$ for $a,b \in R$ and $\alpha \in \Gamma$) such that the following conditions are satisfied for all $a,b,c \in R$:

(i) $a\alpha(b+c)=a\alpha b+a\alpha c$,

(ii) $(a+b)\alpha c=a\alpha c+b\alpha c$,

(iii) $a(\alpha+\beta)c=a\alpha c+a\beta c$,

(iv) $a\alpha(b\beta c)=(a\alpha b)\beta c$.

In the above definition if $(R,+)$ and $(\Gamma,+)$ are canonical hypergroups, then R is called a $\Gamma-$ hyperring. For example, let $(R,+,\bullet)$ be a Krasner hyperring and Γ be an ideal of R. Then R is a $\Gamma-$ hyperring with respect the following hyperoperation:

$x\alpha y = x \bullet \alpha \bullet y$,

where $x, y \in R$ and $\alpha \in \Gamma$.

Suppose that β^* is the fundamental relation on Γ and U_R is the set of all finite sums of elements of R . We define the relation γ as follows:

$a\gamma b \Leftrightarrow \{a,b\} \subseteq u$,

where $u \in U = U_R \cup R\Gamma^{\Sigma} R \cup (U_R + R\Gamma^{\Sigma} R)$.

We denote the transitive closure of γ by γ^*. The equivalence relation γ^* is called fundamental equivalence relation on R. We denoted the equivalence class of the element a by $\gamma^*(a)$. Hence, $\gamma^*(a)=\gamma^*(b)$ if and only if there exist $x_1, x_2, \ldots, x_{n+1}$ with $x_1=a, \ldots, x_{n+1}=b$ and $u_1, \ldots, u_n$ such that $\{x_i, x_{i+1}\} \subseteq u_i$ for $i \in \{1,2,\ldots,n\}$.

Let (R_1,Γ_1) and (R_2,Γ_2) be Γ_1- and Γ_2- semihyperrings, respectively and $f:\Gamma_1 \to \Gamma_2$ be a map. Then $\psi: R_1 \to R_2$ is called a $(\Gamma_1,\Gamma_2)-$homomorphism or shortly homomorphism, if for every $x, y \in R$ and $\alpha \in \Gamma$,

(i) $\psi(x+y)=\{\psi(t) | t \in x+y\} \subseteq \psi(x)+\psi(y)$,

(ii) $\psi(x\alpha y)=\{\psi(t) \mid t \in x\alpha y\} \subseteq \psi(x) f(\alpha) \psi(y)$,

(iii) $f(x+y)=f(x)+f(y)$.

In the above definition if $\psi(x+y)=\psi(x)+\psi(y)$ and $\psi(x\alpha y)=\psi(x) f(\alpha) \psi(y)$, then ψ is called a *strong homomorphism*. The set $\ker\psi=\{(a,b) \in R_1 \times R_2 \mid \psi(a)=\psi(b)\}$ is called the *kernel* of ψ . The homomorphism (ψ, f) is an *epimorphism* if ψ and f are onto and is an isomorphism if ψ and f are isomorphisms.

Proposition 2.4. The relation γ^* is a strongly regular.

Proof. Suppose that $a\gamma^* b$ and x is an arbitrary element of R. It follows that there exist $x_0 = a, x_1, \ldots, x_n = b$ such that for all $i \in \{0,1,\ldots,n-1\}$ we have $x_i \gamma x_{i+1}$. Let $s_1 \in a+x$ and $s_2 \in b+x$. We check that $s_1 \gamma^* s_2$. From $x_i \gamma x_{i+1}$ it follows that there is $u_i \in U$, such that $\{x_i, x_{i+1}\} \subseteq u_i$ and so $\subseteq u_i + x$ and $x_{i+1}+x \subseteq u_i + x$, which means that $(x_i+x)\overline{\gamma}(x_{i+1}+x)$. Hence for all $i \in \{0,1,\ldots,n-1\}$ and for all $t_i \in x_i + x$ we have $t_i \gamma t_{i+1}$. If we consider $t_0 = s_1$, and $t_n = s_2$, then we obtain $s_1\gamma^* s_2$. In the same way, we can prove $(a\alpha x)\overline{\overline{\gamma^*}}(b\alpha x)$ where $\alpha\in\Gamma$. Then γ^* is strongly regular on the right and similarly, it is strongly regular on the left. □

Theorem 2.5. Let R be a $\Gamma-$semihyperring. Then the relation γ^* is the smallest equivalence relation on R such that the quotient R/γ^* is a Γ/β^*-semiring with the following operations:

$$\gamma^*(a)\oplus\gamma^*(b)=\gamma^*(c),\ \text{for some } c\in\gamma^*(a)+\gamma^*(b),$$
$$\gamma^*(a)\odot\beta^*(\gamma)\odot\gamma^*(b)=\gamma^*(d),$$

for some $d\in\gamma^*(a)\beta^*(\gamma)\gamma^*(b)$.

Proof. The proof is straightforward. □

In Theorem 2.5, if R is a $\Gamma-$hyperring, then R/γ^* is a Γ/β^*-ring such that $\gamma^*(0)$ is a zero element of $(R/\gamma^*,\oplus)$ and for every $\gamma^*(a)$, γ^*(-a) is an inverse element of $\gamma^*(a)$.

Let G be the free commutative semigroup generated by $R/\gamma^*\times\Gamma/\beta^*$. We define a relation θ on G as follows:

$$\left(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\right)\theta\left(\prod_{j=1}^{m}(\gamma^*(y_j),\beta^*(\gamma_j))\right),$$

if and only if

$$\bigoplus_{i=1}^{n}\gamma^*(x_i)\odot\beta^*(\alpha_i)\odot\gamma^*(a) =\bigoplus_{j=1}^{m}\gamma^*(y_j)\odot\beta^*(\gamma_j)\odot\gamma^*(a),$$

for all $\gamma^*(a)\in R/\gamma^*$. This relation is a congruence on G. We denote congruence class containing $\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))$ by $\theta\left(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\right)$. Hence G/θ forms a semiring with the following multiplication:

$$\theta\left(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\right)\theta\left(\prod_{j=1}^{m}(\gamma^*(y_j),\beta^*(\gamma_j)\right) =\theta\left(\prod_{i,j}(\gamma^*(x_i)\odot\beta^*(\alpha_i)\odot\gamma^*(y_j),\beta^*(y_j))\right).$$

Obviously, $\theta(\gamma^*(0),\beta^*(\alpha))$ is a zero element of G and $\theta(\prod_{i=1}^{n}\gamma^*(-x_i),\beta^*(\alpha_i))$ is an inverse element of $\theta(\prod_{i=1}^{n}\gamma^*(x_i),\beta^*(\alpha_i))$.

The above semiring is called *fundamental semiring* of $\Gamma-$semihyperring R. If R is a $\Gamma-$hyperring, then G is a ring and is called *fundamental ring* of $\Gamma-$hyperring R. We denote the fundamental semiring by $F(R)$. The category $G\Gamma H$ of all $\Gamma-$semihyperring in which the objects are Γ-semihyperrings, for $\Gamma-$ and $\Gamma-$semihyperrings R_1 and R_2, respectively, $Mor(R_1,R_2)$ is the set of all strong epimorphism from R_1 to R_2. For (R_1,Γ_1), (R_2,Γ_2), (R_3,Γ_3) and $(\varphi_1,f_1)\in Mor(R_1,R_2)$, $(\varphi_2,f_2)\in Mor(R_2,R_3)$, $(\varphi_2,f_2)\circ(\varphi_1,f_1)\in Mor(R_1,R_3)$ denotes the usual composition of maps and is a homomorphism. For every (R,Γ) the map $(I_R,I_\Gamma):R\to R$ is a strong epimorphism and it satisfies $(I_R,I_\Gamma)\circ(\phi,f)=(\phi,f)$ for every $(\varphi,f)\in Mor(R',R)$ and $(\phi,f)\circ(I_R,I_\Gamma)=(\phi,f)$ for every $(\varphi,f)\in Mor(R,R')$. The usual composition of maps satisfies the associative law and is true for homomorphism.

The category SR of all semirings in which the objects are semirings, for semiring R_1 and R_2, $Mor(R_1,R_2)$ is the set of all homomorphism and $I_R:R\to R$ is the usual identity map and it is semiring homomorphism satisfying $\phi\circ I_R=\phi$ for every $\varphi\in Mor(R,R')$ and $I_{R'}\circ\varphi=\varphi$, for $\varphi\in Mor(R',R)$, the composition is the usual composition of homomorphisms.

Theorem 2.6. Let R_1 and R_2 be Γ_1- and Γ_2-semihyperrings, respectively and $(\phi,f):(R_1,\Gamma_1)\to(R_2,\Gamma_2)$ be a strong epimorphism. Then there is a homomorphism $\psi:F(R_1)\to F(R_2)$. Moreover, if (ϕ,f) is an isomorphism, then ψ is an isomorphism.

Proof. We define

$$\psi\left(\theta_1\left(\prod_{i=1}^{n}(\gamma_1^*(x_i),\beta_1^*(\alpha_i))\right)\right) =\theta_2\left(\prod_{i=1}^{n}(\gamma_2^*(\varphi(x_i)),\beta_2^*(f(\alpha_i))\right),$$

first, we prove that ψ is a well-defined. If

$$\theta_1\left(\prod_{i=1}^{n}(\gamma_1^*(x_i),\beta_1^*(\gamma_i)\right)=\theta_1\left(\prod_{j=1}^{m}(\gamma_1^*(y_j),\beta_1^*(\alpha_j)\right),$$

then

$$\oplus_{i=1}^{n}\gamma_1^*(x_i)\odot\beta_1^*(\gamma_i)\odot\gamma_1^*(a) = \oplus_{j=1}^{m}\gamma_1^*(y_j)\odot\beta_1^*(\alpha_j)\odot\gamma_1^*(a),$$

for every $\gamma_1^*(a)\in R_1/\gamma_1^*$.

We define

$$\overline{(\varphi,f)}:(R_1/\gamma_1^*,\Gamma_1/\beta_1^*)\to(R_2/\gamma_2^*,\Gamma_2/\beta_2^*)$$

$$(\gamma_1^*(x),\beta_1^*(\gamma))\mapsto(\gamma_2^*(\varphi(x)),\beta_2^*(f(\gamma)).$$

Let $(\gamma_1^*(x),\beta_1^*(\alpha))=(\gamma_2^*(y),\beta_2^*(\gamma))$. Then there are $y_1,y_2,...,y_{m+1}$ and

$$u_1,u_2,...,u_m\in U_{R_1}\cup(R_1\Gamma^{\Sigma}R_1)\cup(U_{R_1}+R_1\Gamma^{\Sigma}R_1),$$

with $y_1=x$, $y_{m+1}=y$ such that $\{y_i,y_{i+1}\}\subseteq u_i$ for $i\in\{1,2,...,m\}$, which implies that $\{\varphi(y_i),\varphi(y_{i+1})\}\subseteq\varphi(u_i)$. Since ϕ is a homomorphism,

$$\varphi(u_i)\in U_{R_2}\cup(R_2\Gamma^{\Sigma}R_2)\cup(U_{R_2}+R_2\Gamma^{\Sigma}R_2).$$

Hence, $\gamma_2^*(\varphi(\mathrm{x}))=\gamma_2^*(\varphi(\mathrm{y}))$. In the same way, $\beta_2^*(f(\alpha))=\beta_2^*(f(\gamma))$. Thus $\overline{(\phi,f)}$ is well-defined. We have $\overline{(\phi,f)}$ is a homomorphism. Because

$\overline{(\phi,f)}(\gamma_1^*(a)\oplus\gamma_1^*(b))=\overline{(\phi,f)}(\gamma_1^*(c))$ for some $c\in\gamma_1^*(a)+\gamma_1^*(b)$,

we know that $\gamma_1^*(c)=\gamma_1^*(d)$ for some $d\in a+b$. Hence,

$$\begin{aligned}\overline{(\varphi,f)}(\gamma_1^*(a)\oplus\gamma_1^*(b))&=\gamma_2^*(d)\\&=\gamma_2^*(a)\oplus\gamma_2^*(b)\\&=\overline{(\varphi,f)}(\gamma_1^*(a))\oplus\overline{(\varphi,f)}(\gamma_1^*(b)).\end{aligned}$$

In the same way,

$$\overline{(\phi,f)}(\gamma_1^*(a)\odot\beta_1^*(\alpha)\odot\gamma_1^*(b)) = \overline{(\phi,f)}(\gamma_1^*(a))\odot\overline{(\phi,f)}(\beta_1^*(\alpha))\odot\overline{(\phi,f)}(\gamma_1^*(b)).$$

Since $\overline{(\phi,f)}$ is a homomorphism, we have

$$\overline{(\varphi,f)}(\oplus_{i=1}^{n}\gamma_1^*(x_i)\odot\beta_1^*(\gamma_i)\odot\gamma_1^*(a)) = \overline{(\varphi,f)}(\oplus_{j=1}^{m}\gamma_1^*(y_j)\odot\beta_1^*(\alpha_j)\odot\gamma_1^*(a)).$$

Hence,

$$\oplus_{i=1}^{n}\gamma_2^*(\phi(x_i))\odot\beta_2^*(f(\gamma_i))\odot\gamma_2^*(\phi(a)) = \oplus_{j=1}^{m}\gamma_2^*(\phi(y_j))\odot\beta_2^*(f(\alpha_j))\odot\gamma_2^*(\phi(a)).$$

Since ϕ is onto,

$$\theta_2\left(\prod_{i=1}^{n}(\gamma_2^*(\varphi(x_i)),\beta_2^*(f(\gamma_i)))\right) = \theta_2\left(\prod_{j=1}^{m}(\gamma_2^*(\varphi(y_j)),\beta_2^*(f(\alpha_j)))\right).$$

Therefore, ψ is well-defined. We prove that ψ is a homomorphism.

Since, for $d_{i,j}\in x_i\alpha_i y_j$, $\gamma_1^*(c_{i,j})=\gamma_1^*(d_{i,j})$ we have

$$\begin{aligned}\psi\left(\theta_1\left(\prod_{i,j}(\gamma_1^*(c_{i,j}),\beta_1^*(\gamma_j))\right)\right)&=\psi\left(\theta_1\left(\prod_{i,j}(\gamma_1^*(d_{i,j}),\beta_1^*(\gamma_j))\right)\right)\\&=\theta_2\left(\prod_{i,j}(\gamma_2^*(\varphi(d_{i,j})),\beta_2^*(f(\gamma_j)))\right)\end{aligned}$$

We know that $\phi(d_{i,j})\in\phi(x_i\alpha_i y_j)=\phi(x_i)f(\alpha_i)\phi(y_j)\subseteq\gamma_2^*(\phi(x_i))\beta_2^*(f(\alpha_i))\gamma_2^*(\phi(y_j))$. Hence

$$\begin{aligned}&\psi\left(\theta_1\left(\prod_{i=1}^{n}(\gamma_1^*(x_i),\beta_1^*(\gamma_i))\right)\oplus\theta_1\left(\prod_{j=1}^{m}(\gamma_1^*(y_j),\beta_1^*(\gamma_j))\right)\right)\\&=\psi\left(\theta_1\left(\prod_{i=1}^{n}\prod_{j=1}^{m}(\gamma_1^*(x_i),\beta_1^*(\gamma_i))(\gamma_1^*(y_j),\beta_1^*(\gamma_j))\right)\right)\\&=\theta_2\left(\prod_{i=1}^{n}\prod_{j=1}^{m}(\gamma_2^*(\varphi(x_i)),\beta_2^*(f(\gamma_i)))(\gamma_2^*(\varphi(y_j)),\beta_2^*(f(\gamma_j)))\right)\\&=\psi\left(\theta_1\left(\prod_{i=1}^{n}(\gamma_1^*(x_i)),\beta_1^*(\gamma_i))\right)\right)\oplus\psi\left(\theta_1\left(\prod_{j=1}^{m}(\gamma_1^*(y_j),\beta_1^*(\gamma_j))\right)\right).\end{aligned}$$

$$\psi\left(\theta_1\left(\prod_{i=1}^{n}(\gamma_1^*(x_i)),\beta_1^*(\alpha_i))\right)\theta_1\left(\prod_{j=1}^{m}(\gamma_1^*(y_j),\beta_1^*(\gamma_j))\right)\right)$$

$$=\psi\left(\theta_1\left(\prod_{i,j}(\gamma_1^*(x_i)\odot\beta_1^*(\alpha_i)\odot\gamma_1^*(y_j),\beta_1^*(\gamma_j))\right)\right)$$

$$=\psi\left(\theta_1\left(\prod_{i,j}(\gamma_1^*(c_{i,j}),\beta_1^*(\gamma_j))\right)\right)$$

$$=\theta_2\left(\prod_{i,j}(\gamma_2^*(\varphi(d_{i,j})),\beta_2^*(f(\gamma_j)))\right)$$

$$=\theta_2\left(\prod_{i,j}(\gamma_2^*(\varphi(x_i))\odot\beta_2^*(f(\alpha_i))\right.$$

$$\left.\odot\gamma_2^*(\varphi(y_j)),\beta_2^*(f(\gamma_j)))\right)$$

$$=\psi\left(\theta_1\left(\prod_{i=1}^{n}(\gamma_1^*(x_i),\beta_1^*(\alpha_i))\right)\right)$$

$$\psi\left(\theta_1\left(\prod_{j=1}^{m}(\gamma_1^*(y_j),\beta_1^*(\gamma_j))\right)\right),$$

where $c_{i,j}\in\gamma_1^*(x_i)\beta_1^*(\alpha_i)\gamma_1^*(y_j)$.

Hence ψ is a homomorphism. Let (ϕ,f) is an isomorphism. We prove that ψ is an isomorphism. It is enough to we prove that ψ is one-to-one. Suppose that

$$\psi\left(\theta_1\left(\prod_{i=1}^{n}(\gamma_1^*(x_i),\beta_1^*(\alpha_i))\right)\right)$$

$$=\psi\left(\theta_1\left(\prod_{j=1}^{m}(\gamma_1^*(y_j),\beta_1^*(\gamma_j))\right)\right). \quad (*)$$

Firstly, we prove that $\overline{(\phi,f)}$ is one-to-one. Suppose that

$$\overline{(\varphi,f)}(\gamma_1^*(x),\beta_1^*(\alpha))=\overline{(\varphi,f)}(\gamma_1^*(y),\beta_1^*(\gamma)).$$

Then, $\gamma_2^*(\varphi(x))=\gamma_2^*(\varphi(y))$ and $\beta_2^*(f(\alpha))=\beta_2^*(f(\gamma))$. Hence there exist $y_1,y_2,...,y_{m+1}\in R_2$ and

$$u_i\in U_{R_2}\cup R_2\Gamma^{\Sigma}R_2\cup(U_{R_2}+R_2\Gamma^{\Sigma}R_2),$$

for $i\in\{1,2,...,m\}$ such that $y_1=\phi(x)$, $y_{m+1}=\phi(y)$ and

$$\{y_i,y_{i+1}\}\subseteq u_i \quad for \quad i\in\{1,2,...,m\}.$$

Since, ϕ and f is one-to-one and onto, there are $x_i\in R_1$ and

$$v_i\in U_{R_1}\cup R_1\Gamma^{\Sigma}R_1\cup(U_{R_1}+R_1\Gamma^{\Sigma}R_1),$$

such that $\phi(x_i)=y_i$ for $i\in\{1,2,...,m+1\}$ and $\{x_i,x_{i+1}\}\subseteq v_i$ for $i\in\{1,2,...,m\}$. It conclude that $\gamma_1^*(x)=\gamma_1^*(y)$. Similarly, one can see that $\beta_1^*(\alpha)=\beta_1^*(\gamma)$. Therefore $\overline{(\phi,f)}$ is an isomorphism.

By $*$ we have

$$\oplus_{i=1}^{n}\gamma_2^*(\phi(x_i))\odot\beta_2^*(f(\alpha_i))\odot\gamma_2^*(\phi(a))$$

$$=\oplus_{j=1}^{m}\gamma_2^*(\phi(y_j))\odot\beta_2^*(f(\gamma_i))\odot\gamma_2^*(\phi(a)).$$

Hence,

$$\overline{(\phi,f)}(\oplus_{i=1}^{n}\gamma_1^*(x_i)\odot\beta_1^*(\alpha_i)\odot\gamma_1^*(a))$$

$$=\overline{(\phi,f)}(\oplus_{j=1}^{m}\gamma_1^*(y_j)\odot\beta_1^*(\gamma_i)\odot\gamma_1^*(a)),$$

which implies that,

$$\oplus_{i=1}^{n}\gamma_1^*(x_i)\odot\beta_1^*(\alpha_i)\odot\gamma_1^*(a)$$

$$=\oplus_{j=1}^{m}\gamma_1^*(y_j)\odot\beta_1^*(\gamma_i)\odot\gamma_1^*(a).$$

Therefore,

$$\theta_1\left(\prod_{i=1}^{n}(\gamma_1^*(x_i),\beta_1^*(\alpha_i))\right)=\theta_1\left(\prod_{i=1}^{n}(\gamma_1^*(y_j),\beta_1^*(\gamma_j))\right).$$

This implies that ψ is one-to-one. □

Theorem 2.7. Let $G\Gamma H$ be the category of Γ-semihyperrings and SR be the category of semirings. Then there is a covariant functor between $G\Gamma H$ and SR.

Proof. Suppose that R_1 and R_2 be Γ_1- and Γ_2- semihyperrings, respectively. We define, $T(R)=F(R)$ which is a fundamental semiring and $T(\varphi,f)=\psi$, where $(\varphi,f)\in Mor(R_1,R_2)$ and ψ is defined homomorphism in Theorem 2.6. We prove that $T:G\Gamma H\rightarrow SR$ is a covariant functor. Let (ϕ_1,f_1): $(R_1,\Gamma_1)\rightarrow(R_2,\Gamma_2)$ and (ϕ_2,f_2):$(R_2,\Gamma_2)\rightarrow(R_3,\Gamma_3)$ be strong epimorphisms. Let $T(\phi_1,f_1)=\psi_1$ and $T(\varphi_2,f_2)=\psi_2$. We have

$$T\left((\varphi_1,f_1)\circ(\varphi_2,f_2)\right)\left(\theta_1\left(\prod_{i=1}^{n}(\gamma_1^*(x_i),\beta_1^*(\alpha_i))\right)\right)$$

$$=\theta_3\left(\prod_{i=1}^{n}(\gamma_3^*(\varphi_2\circ\varphi_1(x_i)),\beta_3^*(f_2\circ f_1(\alpha_i))\right).$$

Moreover,

$$\psi_2\circ\psi_1\left(\theta_1\left(\prod_{i=1}^{n}(\gamma_1^*(x_i),\beta_1^*(\alpha_i))\right)\right)$$

$$=\psi_2\left(\theta_2\left(\prod_{i=1}^{n}(\gamma_2^*(\varphi_1(x_i)),\beta_2^*(f_1(\alpha_i)))\right)\right)$$

$$=\theta_3\left(\prod_{i=1}^{n}(\gamma_3^*(\varphi_2(\varphi_1(x_i)),\beta_3^*(f_2(f_1(\alpha_i)))\right).$$

Thus, $T((\varphi_2,f_2)\circ(\varphi_1,f_1)) = T(\varphi_2,f_2)\circ T(\varphi_1,f_1)$.

Let $(I_R,I_\Gamma):(R,\Gamma)\to(R,\Gamma)$ be the identity homomorphism. Then

$$T(I_R,I_\Gamma):F(R)\to F(R),$$

is an identity homomorphism. Therefore, T is a covariant functor. □

Proposition 2.8. Let R_1 and R_2 be Γ_1- and Γ_2- semihyperrings, respectively, $(\phi,f):(R_1,\Gamma_1)\to(R_2,\Gamma_2)$ be a strong epimorphism, $\eta_{1,\alpha}:(R_1,\Gamma_1)\to F(R_1)$ and $\eta_{2,\alpha}:(R_1,\Gamma_2)\to F(R_1)$ be maps, where defined by $\eta_{1,\alpha}(x)$ $=\theta_1(\gamma_1^*(x),\beta_1^*(\alpha))$ and $\eta_{1,\alpha}(x)=\theta_1(\gamma_2^*(x),\beta_2^*(\alpha))$. Then there is a homomorphism $\psi:F(R_1)\to F(R_2)$ such that the following diagram is commutative.

$$\begin{array}{ccc}(R_1,\Gamma_1) & \xrightarrow{(\varphi,f)} & (R_2,\Gamma_2)\\ \downarrow & & \downarrow\\ F(R_1) & \xrightarrow{\psi} & F(R_2)\end{array}$$

Proof. Suppose that ψ is the homomorphism defined in Theorem 2.6. We prove that

$$\psi\circ\eta_{1,\alpha}=\eta_{2,f(\alpha)}\circ(\phi,f).$$

Let x be an element of R. Then

$$(\psi\circ\eta_{1,\alpha})(x)=\psi(\eta_{1,\alpha}(x))$$

$$=\psi(\theta_1(\gamma_1^*(x)),\beta_1^*(\alpha))$$

$$=\theta_2(\gamma_2^*(\varphi(x)),\beta_2^*(f(\alpha))),$$

and

$$\eta_{2,f(\alpha)}\circ(\phi,f)(x)=\theta_2(\gamma_2^*(\phi(x)),\beta_2^*(f(\alpha)).$$

Therefore, $\psi\circ\eta_{1,\alpha}=\eta_{2,f(\alpha)}\circ(\varphi,f)$ and this completes the proof. □

Suppose that R is a semiring. A relation ρ on R is called *compatible* if $(a_1,b_1)\in\rho$ and $(a_2,b_2)\in\rho$ imply $(a_1a_2,b_1b_2)\in\rho$ and $(a_1+a_2,b_1+b_2)\in\rho$ for every $a_1,b_1,a_2,b_2\in R$.

A compatible equivalence relation is called *congruence*. If ρ is a congruence on semiring R, then we can define a binary operations on the quotient $R/\rho=\{\rho(x)\mid x\in R\}$ in a natural way as follows:

$$\begin{aligned}\rho(a_1)\oplus\rho(a_2) &= \rho(a_1+a_2),\\ \rho(a_1)\odot\rho(a_2) &= \rho(a_1a_2).\end{aligned}$$

for every $a_1,a_2\in R$.

One can see that the above operations are well-defined and R/ρ is a semiring. Let R and R' be semirings and $\varphi:R\to R'$ be a homomorphism. Then the relation $ker\varphi=\{(a,b)\in R\times R\mid\varphi(a)=\varphi(b)\}$ is congruence on R and there is a monomorphism $\psi:R/ker\varphi\to R'$ such that $Im\psi=Im\varphi$.

Let R be a Γ-semihyperring and ρ be an equivalence relation on R/γ^*. We define

$$\rho^*=\Big\{\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i)\Big),$$

$$\theta\Big(\prod_{j=1}^{m}(\gamma^*(y_j),\beta^*(\alpha_j))\Big)\Big):$$

$$\rho^*=\Big\{\Big(\theta\Big(\prod_{i=1}^{n}\big(\gamma^*(x_i),\beta^*(\alpha_i)\big)\Big),$$

$$\theta\Big(\prod_{j=1}^{m}\big(\gamma^*(y_i),\beta^*(\alpha_i)\big)\Big)\Big):$$

$$\Big(\bigoplus_{i=1}^{n}\gamma^*(x_i)\odot\beta^*(\alpha_i)\odot\gamma^*(y),$$

$$\bigoplus_{j=1}^{m}\gamma^*(y_j)\odot\beta^*(\gamma_j)\odot\gamma^*(y)\Big)$$

$$\in\rho,\ \textit{for every}\ \gamma^*(y)\in R/\gamma^*\Big\}.$$

Proposition 2.9. Let R_1 and R_2 be Γ_1- and Γ_2- semihyperrings and $F(R_1)$, $F(R_2)$ be corresponding semirings, respectively. Suppose that there exists a strong epimorphism (ϕ,f) of the Γ_1-semihyperring R_1

to the Γ_2-semihyperring R_2 and $\overline{(\varphi,f)}$: $(R_1/\gamma_1^*,\Gamma_1/\beta_1^*)\to(R_2/\gamma_2^*,\Gamma_2/\beta_2^*)$. Then,

$$\frac{F(R_1)}{(ker(\overline{(\varphi,f)}))^*}\cong F(R_2).$$

Proof. Let us define a mapping $\psi:F(R_1)\to F(R_2)$ by

$$\psi\left(\theta_1\left(\prod_{i=1}^{n}(\gamma_1^*(x_i),\beta_2^*(\alpha_i))\right)\right)=\theta_2\left(\prod_{i=1}^{n}(\gamma_2^*(\varphi(x_i)),\beta_2^*(f(\alpha_i))\right),$$

for every $\theta_1\left(\prod_{i=1}^{n}(\gamma_1^*(x_i),\beta_2^*(\alpha_i))\right)\in F(R_1)$. By Theorem 2.6, $\overline{(\phi,f)}$ and ψ are homomorphisms. Hence,

$$ker\psi=\Big\{\Big(\theta_1\left(\prod_{i=1}^{n}(\gamma_1^*(x_i),\beta_1^*(\alpha_i))\right),\theta_1\left(\prod_{i=1}^{n}(\gamma_1^*(y_j),\beta_1^*(\gamma_j))\right)\Big):\psi\left(\theta_1\left(\prod_{i=1}^{n}(\gamma_1^*(x_i),\beta_1^*(\alpha_i))\right)\right)=\psi\left(\theta_1\left(\prod_{i=1}^{n}(\gamma_1^*(y_j),\beta_1^*(\gamma_j))\right)\right)\Big\}.$$

This implies that $ker\psi=(ker\overline{(\varphi,f)})^*$. This completes the proof. □

Let R_1 and R_2 be Γ_1- and Γ_2-semihyperrings. Then $R_1\times R_2$ is a $(\Gamma_1,\Gamma_2)-$semihyperring with respect the following hyperoperations:

$$(a_1,b_1)\oplus(a_2,b_2)=\{(z_1,z_2)\mid z_1\in a_1+a_2,z_2\in b_1+b_2\},$$

$$(a_1,b_1)\otimes(\alpha_1,\alpha_2)\otimes(a_2,b_2)=\{(z_1,z_2)\mid z_1\in a_1\alpha_1a_2,z_2\in b_1\alpha_2b_2\}.$$

Proposition 2.10. Let R_1 and R_2 be Γ_1- and Γ_2-semihyperrings, respectively. Then

$$F(R_1\times R_2)\simeq F(R_1)\times F(R_2).$$

Proof. Let γ^*, γ_1^* and γ_2^* be fundamental relations on $R_1\times R_2$, R_1 and R_2, respectively. It is easy to see that

$$\frac{R_1\times R_2}{\gamma^*}\simeq\frac{R_1}{\gamma_1^*}\times\frac{R_2}{\gamma_2^*}.$$

Let G be a free commutative semigroup on $(\frac{R_1}{\gamma_1^*}\times\frac{R_2}{\gamma_2^*})\times(\frac{\Gamma_1}{\beta_1^*}\times\frac{\Gamma_2}{\beta_2^*})$. We define relation θ on G as follows:

$$\left(\prod_{i=1}^{n}((\gamma_1^*(x_i),\gamma_2^*(y_i)),(\beta_1^*(\alpha_i),\beta_2^*(\gamma_i)),\prod_{j=1}^{m}((\gamma_1^*(x'_j),\gamma_2^*(y'_j)),(\beta_1^*(\alpha'_j),\beta_2^*(\gamma'_j)))\right)\in\theta,$$

if and only if

$$\bigoplus_{i=1}^{n}(\gamma_1^*(x_i),\gamma_2^*(y_i))\otimes(\beta_1^*(\alpha_i),\beta_2^*(\gamma_i))\otimes(\gamma_1^*(x),\gamma_2^*(y))=\bigoplus_{j=1}^{m}(\gamma_1^*(x'_j),\gamma_2^*(y'_j))\otimes(\beta_1^*(\alpha'_j),\beta_2^*(\gamma'_j))\otimes(\gamma_1^*(x),\gamma_2^*(y)).$$

For every $(\gamma_1^*(x),\gamma_2^*(y))\in\frac{R_1}{\gamma_1^*}\times\frac{R_2}{\gamma_2^*}$. We define

$$\psi:F(R_1\times R_2)\to F(R_1)\times F(R_2)$$

$$\theta\left(\prod_{i=1}^{n}((\gamma_1^*(x_i),(\gamma_2^*(y_i)),(\beta_1^*(\alpha_i),\beta_2^*(\beta_i))\right)\mapsto\left(\theta_1\left(\prod_{i=1}^{n}(\gamma^*(x_i),\beta_1^*(\alpha_i))\right),\theta_2\left(\prod_{i=1}^{n}(\gamma_2^*(y_i),\beta^*(\beta_i))\right)\right).$$

Obviously, this function is well-defined. We prove that ψ is a homomorphism. We have

$$\psi\left(\theta\left(\prod_{i=1}^{n}((\gamma_1^*(x_i),\gamma_2^*(y_i)),(\beta_1^*(\alpha_i),\beta_2^*(\beta_i))\right)\theta\left(\prod_{j=1}^{m}((\gamma_1^*(x'_j),\gamma_2^*(y'_j)),(\beta_1^*(\alpha'_j),\beta_2^*(\beta'_j))\right)\right)$$

$$=\psi\left(\theta\left(\prod_{i,j}((\gamma^*(x_i)\odot\beta_1^*(\alpha_i))\odot\gamma_1^*(x'_j),\gamma_2^*(y_i)\odot\beta_2^*(\beta_i)\odot\gamma_2^*(y'_j),(\beta_1^*(\alpha'_j),\beta_2^*(\beta'_j))\right)\right)$$

$$=\psi\left(\theta\left(\prod_{i,j}(\gamma_1^*(c_{i,j}),\gamma_2^*(d_{i,j})),(\beta_1^*(\alpha'_j),\beta_2^*(\beta'_j))\right)\right).$$

$$= \psi\left(\theta\left(\prod_{i=1}^{n}(\gamma_1^*(x_i),(\gamma_2^*(y_i)),(\beta_1^*(\alpha_i),\beta^*(\beta_i)\right)\right)$$

$$\psi\left(\theta\left(\prod_{i=1}^{n}(\gamma_1^*(x'_j),(\gamma_2^*(y'_j)),(\beta_1^*(\alpha'_j),\beta^*(\beta'_j)\right)\right),$$

where, $c_{i,j} \in \gamma_1^*(x_i)\beta_1^*(\alpha_i)\gamma_1^*(x'_j)$ and $d_{i,j} \in \gamma_2^*(y_i)$ $\beta_2^*(\beta_i)$ $\gamma_1^*(y'_j)$. Hence ψ is a homomorphism. One can see that ψ is one-to-one and onto and this completes the proof. □

Theorem 2.11. Let R_1, R_2 and R_3 *be* Γ_1-, Γ_2- and Γ_3-hyperrings, respectively and $(\varphi_1,f_1):(R_1,\Gamma_1) \to (R_3,\Gamma_3)$, $(\phi_2,f_2):(R_2,\Gamma_2)\to(R_3,\Gamma_3)$ be strong epimorphisms. Then there is a ring R and $\psi_3: R \to F(R_2)$, $\psi_4: R \to F(R_1)$ such that the following diagrams is commutative. Moreover, if R' is a semiring and $\psi_4': R' \to F(R_1)$, $\psi_4': R' \to F(R_2)$ are homomorphisms such that following diagrams commutative, then there exists a unique homomorphism $\chi: R' \to R$ such that $\psi_4 \circ \chi = \psi_4'$ and $\psi_3 \circ \chi = \psi_3'$.

$$\begin{array}{ccc} R & \xrightarrow{\psi_3} & F(R_2) \\ \downarrow & & \downarrow \\ F(R_1) & \xrightarrow{\psi_1} & F(R_3) \end{array} \qquad \begin{array}{ccc} R' & \xrightarrow{\psi_3} & F(R_2) \\ \downarrow & & \downarrow \\ F(R_1) & \xrightarrow{\psi_1} & F(R_3) \end{array}$$

Proof. Suppose that $F(R_1)$, $F(R_2)$ and $F(R_3)$ are fundamental rings of (R_1,Γ_1), (R_2,Γ_2) and (R_3,Γ_3), respectively. Let $\psi_2: F(R_2)\to F(R_3)$ and $\psi_1: F(R_1) \to F(R_3)$ be homomorphisms defined in Theorem 2.6. Take

$$R = \Big\{\Big(\theta_1\Big(\prod\nolimits_{i=1}^{n}(\gamma_1^*(x_i),\beta_1^*(\alpha_i))\Big),$$
$$\theta_2\Big(\prod\nolimits_{j=1}^{m}(\gamma_2^*(y_j),\beta_2^*(\beta_j))\Big)\Big) \in F(R_1)\times F(R_2) \mid$$
$$\psi_1\Big(\theta_1\Big(\prod\nolimits_{i=1}^{n}(\gamma_1^*(x_i),\beta_1^*(\alpha_i))\Big)\Big)$$
$$= \psi_2\Big(\theta_2\Big(\prod\nolimits_{j=1}^{m}(\gamma_2^*(y_j),\beta_2^*(\beta_j))\Big)\Big)\Big\},$$

and we define $\psi_3: R \to F(R_2)$ and $\psi_4: R \to F(R_1)$ by

$$\psi_4\Big(\theta_1\Big(\prod\nolimits_{i=1}^{n}(\gamma_1^*(x_i),\beta_1^*(\alpha_i))\Big),$$
$$\theta_2\Big(\prod\nolimits_{j=1}^{m}(\gamma_2^*(y_j),\beta_2^*(\beta_j))\Big)\Big)$$
$$= \theta_1\Big(\prod\nolimits_{i=1}^{n}(\gamma_1^*(x_i),\beta_1^*(\alpha_i))\Big),$$

$$\psi_3\Big(\theta_1\Big(\prod\nolimits_{i=1}^{n}(\gamma_1^*(x_i),\beta_1^*(\alpha_i))\Big),$$
$$\theta_2\Big(\prod\nolimits_{j=1}^{m}(\gamma_2(y_j),\beta_2^*(\beta_j))\Big)\Big)$$
$$= \theta_2\Big(\prod\nolimits_{j=1}^{m}(\gamma_2^*(y_j),\beta_2^*(\alpha_j))\Big).$$

The maps ψ_3 and ψ_4 are homomorphisms and $\psi_1 \circ \psi_4 = \psi_2 \circ \psi_3$.

Let R' be a semiring with homomorphism $\psi_3': R' \to F(R_2)$ and $\psi_4': R' \to F(R_1)$ such that $\psi_1 \circ \psi_4' = \psi_2 \circ \psi_3'$. Define $\chi: R' \to R$ by $\chi(x) = (\psi_4'(x), \psi_3'(x))$ $x \in R$. Since, for every $x \in R$, $\psi_1 \circ \psi_4'(x) = \psi_2 \circ \psi_3'(x)$, χ is well-defined. Obviously, χ is a homomorphism. Let $\chi': R' \to R$ be a homomorphism such that $\psi_4 \circ \chi' = \psi_4'$ and $\psi_3 \circ \chi' = \psi_3'$. Then

$$\psi_4 \circ \chi'(x) = \psi_4(a,b) = b,$$

where $\chi'(x) = (a,b)$. Hence $\psi_4'(x) = b$. In the same way, $\psi_3'(x) = a$. Therefore, $\chi = \chi'$. This completes the proof. □

In this paper, we consider $\Gamma-$semihyperrings which is a new kind of hyperalgebra and is a generalization of semihyperrings, hyperrings and rings. Some related properties of $\Gamma-$semihyperrings are described. In particular, we introduce strongly regular relation, fundamental relation and fundamental semiring. The main tools in the theory of hyperstructures are the fundamental relations. By using these concepts, we obtain a covariant functor between the category of $\Gamma-$ semihyperrings and the category of semirings.

Acknowledgements

The authors are highly grateful to referees for their valuable comments and suggestions for improving the paper.

References

1. Ameri, R. On categories of hypergroups and hypermodules. Journal of Discret Mathematics and Cryptology **6**: 121-132 (2003).
2. Anvariyeh, S. M. Mirvakili S. and Davvaz, B. On Γ-hyperideals in Γ-semihypergroups. *Carapathian J. Math.* **26(1)**: 11-23 (2010).
3. Anvariyeh, S .M. Mirvakili, S. and Davvaz, B. θ^*-Relation on hypermodules and fundamental modules over commutative fundamental rings. *Communications in Algebra* **6**(2): 622-631 (2008).
4. Corsini, P. Prolegomena of hypergroup theory. Aviani Editore, Italy, 1993.
5. Corsini, P. and Leoreanu, V. Applications of hyperstructure theory. Advances in Mathematics (Dordrecht), 5, Kluwer Academic Publishers, Dordrecht 2003.
6. Davvaz, B. and Vougiouklis, T. Commutative rings obtained from hyperrings (Hv-rings) with α^*-relations. *Communications in Algebra* **35**: 3307 -3320 (2007).
7. Davvaz, B. and Leoreanu-Fotea, V. Hyperring theory and applications. International Academic Press, USA, 2007.
8. Davvaz, B. and Karimian, M. On the γ^* n-complete hypergroups. *European Journal of Combinatorics* **28**: 86-93 (2007).
9. Freni, D. Une note sur le coeur d'un hypergroupe et sur la cloture β^* de β. *Mat. Pura e Appl* **8**: 153-156 (1991).
10. Freni, D. A new characterization of the derived hyper group via strongly regular equivalences. *Communication in Algebra* **30**(8): 3977-3989 (2002).
11. Freni, D. On a strongly regular relation in hypergroupoids. *Pure Math. Appl, Ser. A* **3-4**: 191-198 (1992).
12. Heidari, D. Dehkordi, S. O. and Davvaz, B. Γ-Semihypergroups and their properties. *U.P.B. Sci. Bull., Series A* **72**: 197-210 (2010).
13. Koskas, M. Groupoides demi-hypergroupes et hypergroupes. J. Math. Pures et Appl. **49**: 155-192 (1970).
14. Marty, F. Sur une g´en´eralization de la notion de groupe. 8th Congress Math. Scandenaves, Stockholm 45-49 (1934).
15. Mirvakili, S. Anvariyeh, S. M. and Davvaz, B. On α-relation and transitivity conditions of α. *Communications in Algebra* **36**: 1695-1703 (2008).
16. Spartalis, S. and Vougiouklis, T. The fundamental relations on Hv-rings, Riv. Mat. Pura Appl. **13**: 7-20 (1994).
17. Prenowitz, W. and Jantosciak, J. Join geometries. Springer-Verlag, UTM, (1979).
18. Vougiouklis, T. Hyperstructures and their representations. Hadronic Press, Inc, Palm Harbor, USA (1994).
19. Vougiouklis, T. The fundamental relation in hyperrings. The general hyperfield, proc. of the 4th Int. Congress on Algebraic Hyperstructures and Appl. (AHA 1990), Xanthi; Greece; World Scientific, 203-211 (1991).
20. Vougiouklis, T. Groups in hypergroups. *Annals Discrete Math.* **37**: 459- 468 (1988).