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Generalized Baer-Invariant of a Pair of Groups and the Direct Limit

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Abstract

In this paper we introduce the concept of generalized Baer-invariant of a pair of groups with respect to two varieties ν and ω of groups. We give some inequalities for the generalized Baer-invariant of a pair of finite groups, when ν is considered to be the Schur-Baer variety. Further, we present a sufficient condition under which the order of the generalized Baer-invariant of a pair of finite groups divides the order of the generalized Baer-invariant of a pair of their factor groups. Moreover, we prove that the generalized Baer-invariant of a pair of groups commutes with direct limit.

Keywords: Verbal and marginal subgroups; Generalized Baer-invariant; Pair of groups; Direct limit

Introduction

Let F_∞ be the free group freely generated by the countable set X={$x_1,x_2,\ldots$} and V be a subset of F_∞. Let ν be the variety of groups defined by the set of laws V. In 1940, P. Hall [1] introduced the following subgroups of a given group G associated with a variety of groups ν as follows:

$$V(G) = <\nu(g_1, g_2, \ldots, g_r) \mid g_i \in G, \nu \in V, 1 \le i \le r>,$$

$$V^*(G) = \{a \in G \mid \nu(g_1, g_2, \ldots, g_i a, \ldots, g_r) = \nu(g_1, g_2, \ldots, g_r); g_i \in G, \nu \in V, 1 \le i \le r\},$$

which are called the verbal and marginal subgroups of G, respectively.

Let N be a normal subgroup of a group G. Then we define $[NV^*G]$ to be the subgroup of G generated by the elements of the following set:

$$\{\nu(g_1, g_2, \ldots, g_i n, \ldots, g_r)\nu(g_1, g_2, \ldots, g_r)^{-1} \mid 1 \le i \le r, \nu \in V, g_1, \ldots, g_r \in G, n \in N\}.$$

It is easily checked that $[NV^*G]$ is the least normal subgroup T of G such that N/T is contained in $V^*(G/T)$ (see [2]). In 1976, Leedham-Green and McKay [5] introduced the following generalized version of the Baer-invariant of a group with respect to two varieties ν and ω.

Let G be an arbitrary group in ω with a free presentation $1 \to R \to F \to G \to 1$, where F is a free group. Clearly, $1 = W(G) = W(F)R/R$ and hence $W(F) \subseteq R$. Thus

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$$1 \to R/W(F) \to F/W(F) \to G \to 1$$

is a $\omega-$*free presentation of the group G. We call*

$$\omega\nu M(G) = \frac{R/W(F) \cap V(F/W(F))}{[R/W(F)V^*(F/W(F)]}$$

$$= \frac{W(F)(R \cap V(F))}{W(F)[RV^*F]}$$

the generalized Baer-invariant of the group G in ω *with respect to the variety* ν. *Now if N is a normal subgroup of the group G for a suitable normal subgroup S of the free group F, we have* $N \cong S/R$. *Then we can define the generalized Baer-invariant of a pair of groups with respect to two varieties* ν *and* ω *as follows:*

$$\omega\nu M(G,N) = \frac{R/W(F) \cap [S/W(F)V^*(F/W(F))]}{[R/W(F)V^*(F/W(F)]}$$

$$= \frac{W(F)(R \cap [SV^*F])}{W(F)[RV^*F]}.$$

One may check that $\omega\nu M(G,N)$ *is always abelian and independent of the free presentation of G. In particular, if* ω *is the variety of all groups and N=G, then the generalized Baer-invariant of the pair (G,N) will be*

$$\nu M(G,G) = \frac{R \cap V(F)}{[RV^*F]} = \nu M(G),$$

which is the usual Baer-invariant of G with respect to ν *(see [5, 9] for more information on the Baer-invariant of groups).*

Results

1. Some Inequalities of the Generalized Baer-Invariant

The following lemma gives some properties of the verbal and marginal subgroups of a group G with respect to a given variety ν .

Lemma 2.1. Let ν be a variety of groups defined by a set of laws V, and N be a normal subgroup of a given group G. Then

(i) $G \in \nu \leftrightarrow V(G) = 1 \leftrightarrow V^*(G) = G$;

(ii) $V(G/N) = V(G)N/N$ and $V^*(G/N) \supseteq V^*(G)N/N$;

(iii) $N \subseteq V^*(G) \leftrightarrow [NV^*G] = 1$;

(iv) $V(N) \subseteq [NV^*G] \subseteq N \cap V(G)$. In particular, $V(G) = [GV^*G]$;

(v) $V(V^*(G)) = 1$ *and* $V^*(G/V(G)) = G/V(G)$.

Proof. It is straightforward. □

The following theorem plays a fundamental role in obtaining the required inequalities.

Theorem 2.2. Let ν and ω be two varieties of groups and G be a group in the variety ω with two normal subgroups K and N such that $K \subseteq N$. Then the following sequence is exact:

(i) $1 \to \omega\nu M(G,K) \to \omega\nu M(G,N) \to$

$$\omega\nu M(G/K, N/K) \to \frac{K \cap [NV^*G]}{[KV^*G]} \to 1.$$

And

(ii) If K is contained in $V^*(G)$, then the following sequence is exact:

$$\omega\nu M(G,N) \to \omega\nu M(G/K, N/K) \to K$$

$$\to \frac{G}{[NV^*G]} \to \frac{G}{[NV^*G]K} \to 1.$$

Proof. Let $1 \to R \to F \to G \to 1$ be a free presentation of G and S, T be normal subgroups of the free group F such that $R \subseteq T \subseteq S$, $N \cong S/R$ and $K \cong T/R$. By the definition

$$\omega\nu M(G,K) = \frac{W(F)(R \cap [TV^*F])}{W(F)[RV^*F]} \quad \omega\nu M(G,N) = \frac{W(F)(R \cap [SV^*F])}{W(F)[RV^*F]},$$

$$\omega\nu M(G/K, N/K) = \frac{W(F)(T \cap [SV^*F])}{W(F)[TV^*F]} \quad \frac{K \cap [NV^*G]}{[KV^*G]} = \frac{(T \cap [SV^*F])R}{[TV^*F]R}.$$

(i) Clearly the following sequence with obvious natural homomorphism is exact:

$$1 \to \frac{W(F)(R \cap [TV^*F])}{W(F)[RV^*F]} \to \frac{W(F)(R \cap [SV^*F])}{W(F)[RV^*F]}$$

$$\to \frac{W(F)(T \cap [SV^*F])}{W(F)[TV^*F]} \to \frac{(T \cap [SV^*F])R}{[RV^*F]R} \to 1.$$

(ii) Using the assumption and Lemma 2.1, $[TV^*F] \subseteq R$. We deduce that inclusion maps

$$W(F)(R \cap [SV^*F]) \to W(F)(T \cap [SV^*F])$$

$$\to T \to F \to F,$$

induce the following exact sequence of homorphisms

$$\frac{W(F)(R\cap[SV*F])}{W(F)[RV*F]}\rightarrow\frac{W(F)(T\cap[SV*F])}{W(F)[TV*F]}$$

$$\rightarrow T/R\rightarrow\frac{F}{[SV*F]R}\rightarrow\frac{F}{[SV*F]T}\rightarrow 1,$$

which gives the result. □

Variety ν is called a Schur-Baer variety if for any group G for which the marginal factor group $G/V(G)$ is finite, then the verbal subgroup $V(G)$ is also finite. In 2002, Moghaddam and others [8] proved that for a finite group G, $\nu M(G)$ and hence $\nu M(G,N)$ are finite, with respect to a Schur-Baer variety ν. We usually work with the varieties having Schur-Baer property. Now with using theorem 2.2, we are able to generalize a result of Jones [3], Moghaddam and others [7] as follows:

Corollary 2.3. Let (G,N) be a pair of finite groups and K be a normal subgroup of G contained in N. Let ν and ω be two varieties of groups such that G be in the variety ω, then

(i) $\left|\frac{K\cap[NV*G]}{[KV*G]}\right||\omega\nu M(G,N)|=|\omega\nu M(G/K,N/K)||\omega\nu M(G,K)|$;

(ii) $d(\omega\nu M(G,N))\le d(\omega\nu M(G/K,N/K))+d(\omega\nu M(G,K))$;

(iii) $e(\omega\nu M(G,N))$ divides $e(\omega\nu M(G/K,N/K))e(\omega\nu M(G,K))$;

(iv) $d(\omega\nu M(G/K,N/K))\le d(\omega\nu M(G,N))+d(\frac{K\cap[NV*G]}{[KV*G]})$;

(v) $e(\omega\nu M(G/K,N/K))\le e(\omega\nu M(G,N))e(\frac{K\cap[NV*G]}{[KV*G]})$.

where $e(X)$ and d(X) are the exponent and the minimal number of generators of the group X, respectively.

Proof. By theorem 2.2, $|\omega\nu M(G,N)|=|L||\omega\nu M(G,K)|$ and

$$\frac{\omega\nu M(G/K,N/K)}{L}\cong\frac{K\cap[NV*G]}{[KV*G]},$$

Where $L=\mathrm{Im}(\omega\nu M(G,N)\rightarrow\omega\nu M(G/K,N/K))$ as in theorem 2.2. So

$$|K\cap[NV*G]||\omega\nu M(G,N)|$$
$$=|K\cap[NV*G]||L||\omega\nu M(G,K)|$$
$$=|[KV*G]||\omega\nu M(G/K,N/K)||\omega\nu M(G,K)|.$$

Hence:

$$\left|\frac{K\cap[NV*G]}{[KV*G]}\right||\omega\nu M(G,N)|$$
$$=|\omega\nu M(G/K,N/K)||\omega\nu M(G,K)|,$$

which prove part (i). Similarly, we can prove (ii), (iii), (iv) and (v). □

Finally, in this section a sufficient condition will be given such that the order of the gene- ralized Baer-invariant of a pair of finite groups divides the order of the generalized Baer-invariant of a pair of their factor groups with respect to two varieties of groups.

Let ν and ω be two varieties of groups defined by sets of laws V and W, respectively. Let E be an arbitrary group and G be a group in ω. Let $\psi:E\rightarrow G$ be an epimorphism such that $Ker\,\psi\subseteq V^*(E)$. We denote by $(WV^*)^*(G)$ the intersection of all subgroups of the form $\psi(V^*(E))$. Clearly, $(WV^*)^*(G)$ is a characteristic subgroup of G which is contained in $V^*(G)$. In particular, if ω is the variety of all groups and ν is a variety of abelian groups, then this subgroup is denoted by $Z^*(G)$ as in [4].

Now using the above concept we have the following theorem.

Theorem 2.4. Let K be a normal subgroup of G contained in $N\cap(WV^*)^*(G)$. Then

$$|\omega\nu M(G,N)|\ divides\ |\omega\nu M(G/K,N/K)|.$$

Proof. By theorem 3.2 of [9], natural homomorphism $\omega\nu M(G)\rightarrow\omega\nu M(G/K)$ will be a monomorphism. Now the following commutative diagram

$$\begin{array}{ccc}\omega\nu M(G,N) & \xrightarrow{\subseteq} & \omega\nu M(G)\\ \downarrow & & \downarrow\\ \omega\nu M(G/K,N/K) & \xrightarrow{\subseteq} & \omega\nu M(G/K)\end{array}$$

implies that the natural homomorphism $\omega\nu M(G,N)\rightarrow\omega\nu M(G/K,N/K)$ is also a monomorphism. Thus theorem 2.2(i) implies that $\omega\nu M(G,K)$ is trivial and hence corollary 2.3(i) gives

$|\omega v M(G/K, N/K)|$
$=|K \cap [NV^*G]|\,|\omega v M(G,N)|$, which completes the result. □

2. Generalized Baer-Invariant and the Direct Limit

Let $\{G_i;\lambda_i^j, I\}$ be a system of groups and I a partially ordered set in such a way that for every $i,j \in I$ there exists $k \in I$ such that $i,j \le k$. For $i \le j$ there exists a homomorphism $\lambda_i^j : G_i \to G_j$ satisfying the following conditions:

(i) For each $i \in I$, $\lambda_i^i : G_i \to G_i$ is the identity homomorphism;

(ii) If $i \le j \le k$, then $\lambda_i^j \circ \lambda_j^k = \lambda_i^k$.

In this case, the system $\{G_i;\lambda_i^j, I\}$ is called a direct system of groups. Let $\bigcup_{i\in I} G_i$ be the disjoint union of groups in the direct system. Then we have an equivalence relation with this set as follows:

$x \sim y$ if and only if $\lambda_i^k(x) = \lambda_j^k(x)$, for some $k \ge i,j$.

Let $\bigcup_{i\in I} G_i/\sim = G$ be the quotient set and denote the equivalence class of an element x by {x}. Now we define the binary operation on G in the following way.

For any {x} and {y} in G there exist $i,j \in I$ such that $x \in G_i$ and $y \in G_j$ then for some $k \ge i,j$,

$$\{x\}\{y\} = \{\lambda_i^k(x)\lambda_j^k(y)\}.$$

Clearly, this operation is well-defined and makes G into a group which is called the direct limit of the direct system $\{G_i;\lambda_i^j, I\}$ and denoted by $G = \varinjlim G_i$.

Two following lemmas are very useful for our investigation, and their proofs are straightforward (see [6, 10]).

Lemma 3.1.

(i) Let $\{G_i;\lambda_i^j, I\}$ be a direct system of groups and $G = \varinjlim G_i$, then G has the universal property;

(ii) The direct system of exact sequences remains exact;

(iii) Every group is the direct limit of its finitely generated subgroups.

Lemma 3.2. Let $\{G_i;\lambda_i^j, I\}$ be a direct system of group and N_i be a normal subgroup of G_i such that $\lambda_i^j(N_i) \subseteq N_j$, for all $i,j \in I\ (i<j)$. Then

(i) $\{N_i;\lambda_i^j|_{N_i}, I\}$ and $\{G_i/N_i;\overline{\lambda_i^j}, I\}$ are both direct system where $\overline{\lambda_i^j}$ is the induced homomorphism;

(ii) If v is the variety of groups and $N_i \subseteq V^*(G_i)$ for all $i \in I$, then $\varinjlim N_i \subseteq V^*(\varinjlim G_i)$;

(iii) $\varinjlim \dfrac{G_i}{N_i} = \dfrac{\varinjlim G_i}{\varinjlim N_i}$.

The following lemma shortens the proof of the main theorem of this section.

Lemma 3.3. Let v be a variety of groups defined by the set laws V and $\{G_i;\lambda_i^j, I\}$ be a direct system of groups. Let $1 \to R_i \to F_i \to G_i \to 1$ be the free presentation of G_i, then

$$\varinjlim [R_iV^*F_i]V(F_i) = [RV^*F]V(F).$$

Proof. If $i \le j$ then there is a homomorphism $\lambda_i^j : G_i \to G_j$ subject to the conditions (i) and (ii) of the definition of direct system. So there exists a homomorphism $\beta_i^j : F_i \to F_j$, where $\beta_i^j = F_i\lambda_i^j$. Also β_i^j restricts to homomorphism, say

$$\alpha_i^j : [R_iV^*F_i]V(F_i) \to [R_jV^*F_j]V(F_j) \quad if \quad i \le j.$$

It is easily verified that $\{[R_iV^*F_i]V(F_i);\alpha_i^j, I\}$ is a direct system and hence we have direct limit denoted by $\varinjlim [R_iV^*F_i]V(F_i)$. If $b \in \varinjlim [R_iV^*F_i]V(F_i)$ then there exists $i \in I$ such that $\lambda_i(b_i) = b$, where $b_i \in [R_iV^*F_i]V(F_i)$. Hence $b_i = c_id_i$, where $c_i \in [R_iV^*F_i]$ and $b_i \in V(F_i)$. So $b = \lambda_i(c_i)\lambda_i(d_i) \in \varinjlim [R_iV^*F_i] \varinjlim V(F_i)$. Now if $a \in \varinjlim [R_iV^*F_i] \varinjlim V(F_i)$ then $a = \lambda_{i_1}(b_{i_1})\lambda_{i_2}(c_{i_2})$. Also, there exists $j \in I$ Such that $a = \lambda_j(b_{i_1})\lambda_j(c_{i_2})$. Thus, $\varinjlim [R_iV^*F_i] \varinjlim V(F_i) \subseteq \varinjlim [R_iV^*F_i]V(F_i)$, Also, by lemma 3.4 of [6], $\varinjlim [R_iV^*F_i] = [RV^*F]$ and $\varinjlim V(F_i) = V(F)$. Therefore,

$$\varinjlim [R_iV^*F_i]V(F_i) = \varinjlim [R_iV^*F_i] \varinjlim V(F_i)$$
$$= [RV^*F]V(F). \ \square$$

In 2004, Moghaddam et al. proved the following (see [10]).

Theorem 3.4. Let $\{G_i;\lambda_i^j, I\}$ be a direct system of groups in the variety ω. Then for any variety v of groups, the generalized Baer-invariant commutes with the direct limit, that is

$$\omega\nu M(\varinjlim G_i) = \varinjlim \omega\nu M(G_i).$$

Now we prove the following result which generalizes theorem 3.4, extensively.

Theorem 3.5. Let ν and ω be two varieties of groups and $\{G_i;\lambda_i^j,I\}$ be a direct system of groups in the variety ω. Let N_i be a normal subgroup of G_i such that $\lambda_i^j(N_i)\subseteq N_j$, for all $i,j\in I\ (i<j)$, then

$$\omega\nu M(\varinjlim G_i, \varinjlim N_i) = \varinjlim \omega\nu M(G_i, N_i).$$

Proof. Let $1\to R_i\to F_i\to G_i\to 1$ be the free presentation of G_i where F_i is the free group on the underlying set G_i. Then there exist homomorphism $\beta_i^j: F_i\to F_j$ such that the following diagram is commutative

$$\begin{array}{ccccccccc}
1\to & R_i & \to & F_i & \to & G_i & \to 1 \\
 & \downarrow \beta_i^j| & & \downarrow \beta_i^j & & \downarrow \lambda_i^j & \\
1\to & R_j & \to & F_j & \to & G_j & \to 1.
\end{array}$$

Let S_i be a normal subgroup of F_i such that $S_i/R_i\cong N_i$, then it is clear that $\{F_i;\beta_i^j,I\}$, $\{S_i;\beta_i^j\|,I\}$ and $\{R_i;\beta_i^j|,I\}$ are direct system of groups. Put $F=\varinjlim F_i$, $S=\varinjlim S_i$ and $R=\varinjlim R_i$. One observes that $\{R_i\cap[S_iV^*F_i];\beta_i^j|,I\}$ and $\{[R_iV^*F_i];\beta_i^j|,I\}$ are direct system of groups, where $\beta_i^j|$ is the restriction of β_i^j to the related subgroups. Also, $\{W(F_i)[R_iV^*F_i];\beta_i^j|,I\}$ and $\{W(F_i)(R_i\cap [S_iV^*F_i]);\ \beta_i^j|,I\}$ are direct systems of groups. Now by lemma 3.3 we have $\varinjlim[R_iV^*F_i]V(F_i) = [RV^*F]V(F)$ and $\varinjlim W(F_i)(R_i\cap[S_iV^*F_i]) = \varinjlim W(F)(R\cap[SV^*F])$. Now the definition gives the exactness of the following sequence

$$\begin{aligned}
1&\to W(F_i)[R_iV^*F_i] \\
&\to W(F_i)(R_i\cap[S_iV^*F_i]) \\
&\to \omega\nu M(G_i,N_i)\to 1.
\end{aligned}$$

Lemma 3.1. (ii) implies that the following sequence is also exact

$$\begin{aligned}
1&\to \varinjlim W(F_i)[R_iV^*F_i] \\
&\to \varinjlim W(F_i)(R_i\cap[S_iV^*F_i]) \\
&\to \varinjlim \omega\nu M(G_i,N_i)\to 1.
\end{aligned}$$

Hence $\varinjlim \omega\nu M(G_i,N_i) = \omega\nu M(\varinjlim G_i, \varinjlim N_i)$. □

The following corollary gives generalized version of the direct limit of exact sequences in theorem 2.2.

Corollary 3.6. Let ν and ω be two varieties of groups and $\{G_i;\lambda_i^j,I\}$ be a direct system of groups in the variety ω. Let N_i and K_i be normal subgroups of G_i such that $K_i\subseteq V^*(G_i)$, $\lambda_i^j(N_i)\subseteq N_j$ and $\lambda_i^j(K_i)\subseteq K_j$ for all $i,j\in I\ (i<j)$. If $G=\varinjlim G_i$, $N=\varinjlim N_i$ and $K=\varinjlim K_i$ then the following sequences are exact:

(i) $1\to \omega\nu M(G,K)\to \omega\nu M(G,N)$

$$\to \omega\nu M(G/K, N/K)\to \frac{K\cap[NV^*G]}{[KV^*G]}\to 1;$$

(ii) $\omega\nu M(G,N)\to \omega\nu M(G/K, N/K)$

$$\to K\to \frac{G}{[NV^*G]}\to \frac{G}{[NV^*G]K}\to 1.$$

Proof. Since $K_i\subseteq V^*(G_i)$ for all $i\in I$, then Lemma 3.2 (ii) giva $K\subseteq V^*(G)$. and Lemma 3.1(ii) give the exactness of the following sequences:

(i) $1\to \varinjlim \omega\nu M(G_i,K_i)\to \varinjlim \omega\nu M(G_i,N_i)$

$$\to \varinjlim \omega\nu M(G_i/K_i, N_i/K_i)$$

$$\to \varinjlim \frac{K_i\cap[N_iV^*G_i]}{[K_iV^*G_i]}\to 1.$$

(ii) $\varinjlim \omega\nu M(G_i,N_i)\to \varinjlim \omega\nu M(G_i/K_i, N_i/K_i)$

$$\to \varinjlim K_i\to \varinjlim \frac{G_i}{[N_iV^*G_i]}$$

$$\to \varinjlim \frac{G_i}{[N_iV^*G_i]K_i}\to 1.$$

for all $i\in I$. Now Lemma 3.2(iii) and Theorem 3.5 give the result. □

Discussion

Theorem 3.5 has application that if one works with the generalized Baer-invariant of a pair of groups, one only needs to consider the pair to be finitely generated. As we refer to Lemma 3.1 (iii), it is obvious that every group is the direct limit of its finitely generated subgroups (see [6]).

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