Journal of Sciences, Islamic Republic of Iran 22(2): 153-157 (2011) http://jsciences.ut.ac.ir
University of Tehran, ISSN 1016-1104

Lower Bounds of Copson Type for Hausdorff Matrices on Weighted Sequence Spaces

R. Lashkaripour* and G. Talebi

Department of Mathematics, Faculty of Mathematics, University of Sistan and Baluchestan, Zahedan, Islamic Republic of Iran

Received: 2 December 2009 / Revised: 3 August 2011 / Accepted: 12 August 2011

Abstract

Let $H=(h_{n,k})_{n,k\geq 0}$ be a non-negative matrix. Denote by $L_{w,p,q}(H)$, the supremum of those L, satisfying the following inequality:

$$\left(\sum_{n=0}^{\infty} w_n\left(\sum_{k=0}^{\infty} h_{n,k}x_k\right)^q\right)^{\frac{1}{q}} \geq L\left(\sum_{k=0}^{\infty} w_k x_k^p\right)^{\frac{1}{p}},$$

where $x\geq 0$, $x\in l_p(w)$, and also $w=(w_n)$ is increasing, non-negative sequence of real numbers. If $p=q$, we used $L_{w,p}(\mathrm{H})$, instead of $L_{w,p,p}(\mathrm{A})$. The purpose of this paper is to establish a Hardy type formula for $L_{w,p,q}(H_\mu)$, where H_μ is Hausdorff matrix and $0<q\leq p<1$. A similar result is also established for $L_{w,p,q}(H_\mu^t)$ where $-\infty<q\leq p<0$. In particular, we apply our results to the Cesaro matrices, Holder matrices and Gamma matrices. Our results also generalize some works due to R. Lashkaripour and D. Foroutannia [6]. Moreover, in this study we extend some results mentioned in [3] and [4].

Keywords: Lower bound; Weighted sequence space; Lower triangular matrix

Introduction

Let $p\in\mathbb{R}\setminus\{0\}$ and also let $l_p(w)$ denote the space of all real sequences $x=\{x_k\}_{k=0}^{\infty}$ such that

$$\|x\|_{w,p} := \left(\sum_{k=1}^{\infty} w_k x_k^p\right)^{1/p} < \infty,$$

where $w=(w_n)_{n=0}^{\infty}$ is an increasing, non-negative sequence of real numbers with $w_0=1$. We write $x\geq 0$ if $x_k\geq 0$ for all k. We also write $x\uparrow$ for the case that $x_0\leq x_1\leq\ldots\leq x_n\leq\ldots$. The symbol $x\downarrow$, is defined in a similar way. For $p,q\in\mathbb{R}\setminus\{0\}$, the lower bound involved here is the number $L_{w,p,q}(\mathrm{H})$, which is defined as the supremum of those L obeying the

2000 Mathematics Subject Classification: 26D15; 47A30; 40G05; 47D37; 46A45; 54D55.

* Corresponding author, Tel.: :+98(541)2447166, Fax: :+98(541)2447166, E-mail: lashkari@hamoon.usb.ac.ir

following inequality:

$$\left(\sum_{n=0}^{\infty} w_n \left(\sum_{k=0}^{\infty} h_{n,k} x_k\right)^q\right)^{\frac{1}{q}} \geq L \left(\sum_{k=0}^{\infty} w_k x_k^p\right)^{\frac{1}{p}},$$

$$\left(x \geq 0, x \in l_p(w)\right),$$

where $H \geq 0$, that is $H = (h_{n,k})_{n,k\geq 0}$ is a non-negative matrix. We have

$$L_{w,p,q}(\mathrm{H}) \leq \| H \|_{w,p,q},$$

We are interested in the problem of finding the exact value of $L_{w,p,q}(\mathrm{H})$ for the cases: $H = H_\mu$ or $H = H_\mu^t$, where $d\mu$ is a Borel probability measure on [0,1], $(.)^t$ denotes the transpose of (.) and $H_\mu = (h_{n,k})_{n,k\geq 0}$ is the Hausdorff matrix associated with $d\mu$, defined by

$$h_{n,k} = \begin{cases} \binom{n}{k} \int_0^1 \theta^k (1-\theta)^{n-k} \, d\mu(\theta) & n \geq k \\ 0 & n < k. \end{cases}$$

Clearly, $h_{n,k} = \binom{n}{k} \Delta^{n-k} \mu_k$ for $n \geq k \geq 0$, where

$$\mu_k = \int_0^1 \theta^k d\mu(\theta) \qquad (k = 0,1,...)$$

and $\Delta\mu_k = \mu_k - \mu_{k+1}$.

In ([6], Corollary 4.3.3) the author obtained $L_{w,p}\left(\mathrm{C}(1)^t\right) = p$, $0 < p < 1$, where $\mathrm{C}(1) = (c_{n,k})_{n,k\geq 0}$ is the Cesaro matrix defined by:

$$c_{n,k} = \begin{cases} \dfrac{1}{n+1} & 0 \leq k \leq n \\ 0 & ow. \end{cases}$$

This is analogue of Copson results [5, Eq(1.1)](see also[7], Theorem 344) for weighted sequence space $l_p(w)$ and has been generalized by D. Foroutannia. He extended it in ([6], Theorem 2.7.17 and Theorem 2.7.19) to those summability matrices H, whose rows are increasing or decreasing. Also, he gave upper bounds or lower bounds to $L_{w,p}(H)$ for such H. For the case of Hausdorff matrices, the related result with $0 < p < 1$ have been established in [6, Theorem 4.3.2 and Theorem 4.3.7], where the author prove that

$$L_{w,p}(H_\mu^t) = \int_0^1 \theta^{-1/p^*} d\mu(\theta) \qquad (0 < p \leq 1),$$

and

$$L_{w,p}(H_\mu) = \int_0^1 \theta^{-1/p} d\mu(\theta) \qquad (-\infty < p < 0),$$

where $\dfrac{1}{p} + \dfrac{1}{p^*} = 1$.

The exact value of $L_{w,p}(H_\mu)(0 < p \leq 1)$ and $L_{w,p}(H_\mu^t)(-\infty < p < 0)$ have not been found yet. In This paper, we shall fill in this gap. The details are described below.

Results

1. Lower Bounds for Hausdorff Matrices

The purpose of this section is to prove that

$$L_{w,p,q}(H_\mu) \geq \int_{(0,1]} \theta^{-1/q} \, d\mu(\theta) \; (0 < q \leq p \leq 1) \qquad (1\text{-}1)$$

and

$$L_{w,p,q}(H_\mu^t) \geq \int_{(0,1]} \theta^{-1/p^*} d\mu(\theta) (-\infty < q \leq p < 0), \; (1\text{-}2)$$

(see Theorem 1.4 and Theorem 1.5).

Lemma 1.1. Let $0 < p < 1$ and let A be a lower triangular matrix with non-negative entries. If

$$\sup_{n\geq 0} \sum_{k=0}^{n} a_{n,k} = R,$$

and

$$\inf_{k\geq 0} \sum_{n=k}^{\infty} a_{n,k} = C > 0,$$

then $\|Ax\|_{w,p} \geq L \|x\|_{w,p}$ with

$$L \geq R^{\frac{1}{p^*}} C^{\frac{1}{p}}. \qquad (1\text{-}3)$$

Proof. Since (w_n) is increasing, we have $\|Ax\|_{w,p} \geq \|Ax\|_{p,p}$. The desire inequality now is a consequence of ([2], Proposition 7.4). □

For $\alpha \geq 0$, let $E(\alpha) = (e_{n,k}(\alpha))_{n,k} \geq 0$ denote the Euler matrix, defined by:

$$e_{n,k} = \begin{cases} \binom{n}{k} \alpha^k (1-\alpha)^{n-k} & n \geq k \\ 0 & n < k. \end{cases}$$

(cf. [1, p.410]). For $\Omega \subset (0,1],$ we have

$$\int_{\Omega} e_{n,k}(\theta) d\mu(\theta) = \mu(\Omega) \times \int_0^1 e_{n,k}(\theta) d\nu(\theta),$$

where $d\nu = \frac{\chi_\Omega}{\mu(\Omega)} d\mu$ is a Borel probability measure on [0,1] with $\nu(\{0\}) = 0.$ Hence the second part of ([2], Proposition 19.2) can be generalized in the following way.

Lemma 1.2. Let $0 < p \le 1,$ $\Omega \subseteq [0,1]$ and $d\mu$ be any Borel probability measure on [0,1].

If $\mu(\{0\}) = 0$ or $\Omega \subset (0,1],$ then $\left\| \left\{ \int_{\Omega} e_{n,k}(\theta) d\mu(\theta) \right\}_{n=k}^{\infty} \right\|_{w,p}$ increases with k.

Proof. Applying Lemma 1.1 and ([2], Proposition 19.2), we have the statement. □

Lemma 1.3. Let $0 < p \le 1$. Then $L_{w,p}\left(E(\alpha)\right) \ge \alpha^{-1/p}$ for $0 < \alpha \le 1.$

Proof. We have $\sum_{k=0}^{\infty} e_{n,k}(\alpha) = 1 (n \ge 0)$ and $\sum_{n=0}^{\infty} e_{n,k}(\alpha) = \alpha^{-1} (k \ge 0).$ Applying Lemma 1.1 to the case that R = 1 and C = α^{-1} we deduce that $L_{w,p}\left(E(\alpha)\right) \ge \alpha^{-1/p}$ for $0 < p \le 1$. For $p = 1$ from Fubini's theorem and monotonocity of (w_n), we deduce that

$$\begin{aligned} \|E(\alpha)x\|_{w,1} &= \sum_{n=0}^{\infty} w_n \left(\sum_{k=0}^{\infty} e_{n,k}(\alpha) x_k \right) \\ &\ge \sum_{k=0}^{\infty} w_k \left(\sum_{n=0}^{\infty} e_{n,k}(\alpha) \right) x_k \\ &= \alpha^{-1} \|x\|_{w,1} \qquad (x \ge 0) \end{aligned}$$

which gives the desired inequality. This completes the proof. □

Now, we try to establish (1-1) and its related properties. For $x \ge 0,$ we have $H_\mu x = \int_0^1 E(\theta) x \, d\mu(\theta).$ Hence Lemma 1.3 enables us to estimate the value of $L_{w,p,q}\left(H_\mu\right).$ Our results are stated below.

Theorem 1.4. We have

$$L_{w,p,q}\left(H_\mu\right) \ge \int_{(0,1]} \theta^{-1/q} d\mu(\theta) \quad (0 < q \le p \le 1). \quad (1\text{-}4)$$

Moreover, for $0 < q < p \le 1,$ (1-4) is an equality if and only if $\mu(\{0\}) + \mu(\{1\}) = 1$ or the right side of (1-4) is infinity.

Proof. Consider (1-4), let $x \ge 0,$ $\|x\|_{w,p} = 1.$ Then $\|x\|_{w,q} \ge \|x\|_{w,p} = 1.$ Applying Minkowski's inequality and Lemma 1.3, we obtain

$$\begin{aligned} \|H_\mu x\|_{w,q} &= \left\| \int_0^1 E(\theta) x \, d\mu(\theta) \right\|_{w,q} \\ &\ge \int_{(0,1]} \|E(\theta) x\|_{w,q} \, d\mu(\theta) \\ &\ge \left(\int_{(0,1]} \theta^{-1/q} d\mu(\theta) \right) \|x\|_{w,q} \\ &\ge \int_{(0,1]} \theta^{-1/q} d\mu(\theta) \end{aligned}$$

This leads us to (1-4).

Obviously, (1-4) is an equality if its right side is infinity. For the case that $\mu(\{0\}) + \mu(\{1\}) = 1,$ we have

$$\begin{aligned} \|H_\mu e_1\|_{w,q} &= \left(\sum_{n=1}^{\infty} w_n h_{n,1}^q \right)^{\frac{1}{q}} \\ &= \left(\sum_{n=1}^{\infty} w_n \left(\binom{n}{1} \int_0^1 \theta (1-\theta)^{n-1} d\mu(\theta) \right)^q \right)^{\frac{1}{q}} \\ &= \mu(\{1\}) = \int_{(0,1]} \theta^{-1/q} d\mu(\theta), \qquad (1-5) \end{aligned}$$

where $e_1 = (0,1,0,0,...)$. This follows that

$$L_{w,p,q}\left(H_\mu\right) \le \int_{(0,1]} \theta^{-1/q} d\mu(\theta)$$

and consequently, (1-4) is an equality.

Consequently, let $0 < q < p \le 1$ and assume that $\mu(\{0\}) + \mu(\{1\}) \ne 1$ and also that

$$\int_{(0,1]} \theta^{-1/q} d\mu(\theta) < \infty,$$

then $\mu((0,1)) \ne 0$. Since $0 < q < 1,$ we have

$$\sum_{n=0}^{\infty}(1-\theta)^n < \sum_{n=0}^{\infty}(1-\theta)^{nq}. \qquad \theta \in (0,1) \tag{1-6}$$

Applying (1-6), Minkowski's inequality and monotonicity of w we have:

$$\int_{(0,1]} \theta^{-1/q} d\mu(\theta) = \int_{(0,1]} \left(\sum_{n=0}^{\infty}(1-\theta)^n\right)^{\frac{1}{q}} d\mu(\theta)$$

$$< \int_{(0,1]} \left(\sum_{n=0}^{\infty}(1-\theta)^{nq}\right)^{\frac{1}{q}} d\mu(\theta)$$

$$\leq \left\| \left\{ \int_{(0,1]} (1-\theta)^n d\mu(\theta) \right\}_{n=0}^{\infty} \right\|_q \tag{1-7}$$

$$\leq \left\| \left\{ \int_{(0,1]} (1-\theta)^n d\mu(\theta) \right\}_{n=0}^{\infty} \right\|_{w,q}.$$

From (1-7) we can find β satisfying $0<\beta<1$ such that

$$\int_{(0,1]} \theta^{-1/q} d\mu(\theta) < \beta \left\| \left\{ \int_{(0,1]} (1-\theta)^n d\mu(\theta) \right\}_{n=0}^{\infty} \right\|_{w,q}. \tag{1-8}$$

We claim that

$$L_{w,p,q}(H_\mu) \geq \min\left(\beta^{\frac{q-p}{p}} \int_{(0,1]} \theta^{-1/q} d\mu(\theta), \right.$$

$$\left. \beta \left\| \left\{ \int_{(0,1]} (1-\theta)^n d\mu(\theta) \right\}_{n=0}^{\infty} \right\|_{w,q} \right). \tag{1-9}$$

Let $x \geq 0$, with $\|x\|_{w,p} = 1$. We divide the proof into two cases: $x_{k_0} \geq \beta$ for some k_0 or $x_k < \beta$ for all k. For the first case, it follows from Lemma 1.2 that

$$\|H_\mu x\|_{w,q} \geq x_{k_0} \left(\sum_{n=0}^{\infty} w_n h_{n,k_0}^q\right)^{\frac{1}{q}}$$

$$\geq \beta \left\| \left\{ \int_{(0,1]} e_{n,k_0}(\theta) d\mu(\theta) \right\}_{n=k_0}^{\infty} \right\|_{w,q}$$

$$\geq \beta \left\| \left\{ \int_{(0,1]} e_{n,0}(\theta) d\mu(\theta) \right\}_{n=0}^{\infty} \right\|_{w,q}$$

$$= \beta \left\| \left\{ \int_{(0,1]} (1-\theta)^n d\mu(\theta) \right\}_{n=0}^{\infty} \right\|_{w,q}$$

As for the second case, we have $x_k^q \geq \beta^{q-p} x_k^p$ for all k. This implies

$$\sum_{k=0}^{\infty} w_k x_k^q \geq \beta^{q-p} \sum_{k=0}^{\infty} w_k x_k^p = \beta^{q-p}.$$

Applying (1-4), we deduce that

$$\|H_\mu x\|_{w,q} \geq \left(\int_{(0,1]} \theta^{-1/q} d\mu(\theta) \right) \|x\|_{w,p}$$

$$\geq \beta^{\frac{q-p}{p}} \int_{(0,1]} \theta^{-1/q} d\mu(\theta).$$

Hence, no matter which case occurs, $\|H_\mu x\|_{w,q}$ is always greater than or equal to the minimum stated at the right side of (1-9). This leads us to (1-9). It is clear that $\beta^{\frac{q-p}{q}} > 1$. Putting (1-8) and (1-9) together, we have

$$L_{w,p,q}(H_\mu) > \int_{(0,1]} \theta^{-1/q} d\mu(\theta).$$

This completes the proof. □

For $-\infty < q \leq p < 0$, we have $0 < p^* \leq q^* < 1$ where $\frac{1}{p} + \frac{1}{p^*} = 1$ and $\frac{1}{q} + \frac{1}{q^*} = 1$. Applying ([8], Proposition 2.7), $L_{w,p,q}(H_\mu^t) = L_{w,q^*,p^*}(H_\mu)$. Putting this with Theorem 1.4, we get the following result.

Theorem 1.5. Let $\frac{1}{p} + \frac{1}{p^*} = 1$. Then

$$L_{w,p,q}(H_\mu^t) \geq \int_{(0,1]} \theta^{-1/p^*} d\mu(\theta) (-\infty < q < p \leq 0). \tag{1-10}$$

Moreover, for $-\infty < q < p \leq 0$, (1-10) is an equality if and only if $\mu(\{0\}) + \mu(\{1\}) = 1$ or the right side of (1-10) is infinity.

2. Particular Cases

In the following, we present several special cases of Theorems 2.1 and 2.2. Let $d\mu(\theta) = \alpha(1-\theta)^{\alpha-1} d\mu(\theta)$, where $\alpha > 0$. Then $H\mu$ reduces to the Cesaro matrix

$C(\alpha)$ (see [1, p.410]). For $0 < p \le 1,$ we have

$$\int_{(0,1]} \theta^{-1/q} d\,\mu(\theta) = \alpha \int_{(0,1]} \theta^{-1/q} (1-\theta)^{\alpha-1} d\,\theta = \infty.$$

Similarly

$$\int_{(0,1]} \theta^{-1/p^*} d\,\mu(\theta) = \infty. \qquad (-\infty < p < 0)$$

Applying (1-4) and (1-10), we get the following results.

Corollary 2.1. Let $\alpha > 0.$ Then $L_{w,p,q}\left(C(\alpha)\right) = \infty$ for $0 < q \le p \le 1.$ Also we have $L_{w,p,q}\left(C(\alpha)^t\right) = \infty$ for $-\infty < q \le p < 0.$

Next, consider the case $d\,\mu(\theta) = \dfrac{\left|\log\theta\right|^{\alpha-1}}{\Gamma(\alpha)} d\,\theta,$ where $\alpha > 0.$ For this case, H_μ reduces to the

Holder matrix $H(\alpha)$ (see [1, p.410]). We have

$$\int_{(0,1]} \theta^{-1/q} d\,\mu(\theta) = \infty \qquad (0 < q \le 1),$$

and

$$\int_{(0,1]} \theta^{-1/p^*} d\,\mu(\theta) = \infty \qquad (-\infty < p < 0).$$

Hence, the following is a consequence of (1-4) and (2-10).

Corollary 2.2. Let $\alpha > 0$. Then $L_{w,p,q}\left(H(\alpha)\right) = \infty$ for $0 < q \le p \le 1.$ Also, we have $L_{w,p,q}\left(H(\alpha)^t\right) = \infty$ for $-\infty < q \le p < 0.$

The third special case that we consider is $d\,\mu(\theta) = \alpha\theta^{\alpha-1} d\,\theta,$ where $\alpha > 0.$ Then H_μ becomes the Gamma matrix $\Gamma(\alpha)$)(see [1, p.410]). We have

$$\int_{(0,1]} \theta^{-1/q} d\,\mu(\theta) = \alpha \int_{(0,1]} \theta^{-1/q+\alpha-1} d\,\mu(\theta)$$

$$= \begin{cases} \infty & \alpha \le 1/q \\ \dfrac{\alpha}{\alpha - 1/q} & \alpha > 1/q. \end{cases} \qquad \text{(2-1)}$$

Applying Theorem 1.4, we get the following corollary.

Corollary 2.3. Let $\alpha > 0$ and $0 < q \le p \le 1.$ Then $L_{w,p,q}\left(\Gamma(\alpha)\right) = \infty,$ for $\alpha \le 1/q.$ Also, $L_{w,p,q}\left(\Gamma(\alpha)\right) \ge \dfrac{\alpha}{\alpha - 1/q}$ for $\alpha > 1/q.$

Replace q in (2-1) by p^*. Then Theorem 1.5 gives the following consequence.

Corollary 2.4. Let $\alpha > 0,$ $-\infty < q \le p < 0$ and $\dfrac{1}{p} + \dfrac{1}{p^*} = 1.$ Then $L_{w,p,q}\left(\Gamma(\alpha)^t\right) = \infty,$ for $\alpha \le 1/p^*.$ Also, $L_{w,p,q}\left(\Gamma(\alpha)^t\right) \ge \dfrac{\alpha}{\alpha - 1/q}$ for $\alpha > 1/p^*.$

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