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Optimum Block Size in Separate Block Bootstrap to Estimate the Variance of Sample Mean for Lattice Data

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Abstract

The statistical analysis of spatial data is usually done under Gaussian assumption for the underlying random field model. When this assumption is not satisfied, block bootstrap methods can be used to analyze spatial data. One of the crucial problems in this setting is specifying the block sizes. In this paper, we present asymptotic optimal block size for separate block bootstrap to estimate the variance of sample mean for spatial lattice data, using minimization of asymptotic mean square error of the estimator. Further, an empirical method has been proposed to determine the optimal block size. Also the optimality of the empirical estimate of block size has been considered numerically in a simulation study.

Keywords: Lattice data; Separate block bootstrap; Block size; α-Mixing

Introduction

Statistical methods are frequently based on independent observations; however, we are often faced with many cases in which the data depend on each other. Spatial data are observations where their dependency is derived from their location in the space under study. This dependency is described as a function of the distances between the locations of observations. Inferences of spatial data are often based on the assumption of a Gaussian random field, although it may be inappropriate in many practical applications. Specifying correlation structure in spatial statistics may face some problems from the estimation point of view. In such a case, bootstrap method can be used in a nonparametric inference for data.

Efron [5] proposed the bootstrap method for independent data in which one can estimate the bias, variance and the distribution of the estimators using resampling data. This method is not applicable to dependent data such as time series and spatial data (see e.g. Singh [17]). In such cases, block bootstrap methods can be used. Hall [6] proposed two methods based on making observations and locations as blocks for the special case of mosaic data. Buhlmann and Kunsch [2], Zhu and Lahiri [18] and Lahiri [12] also proposed the moving block bootstrap (MBB) method for spatial data analysis. In this method, resampling from observations is performed in moving blocks, however, the observations located at the edges of the study region are less likely to be present in blocks leading to bias in the estimation. To overcome this difficulty, Iranpanah and Mohammadzadeh [8,9] proposed separate block bootstrap (SBB) method to estimate precision measures of the estimators for a random field mean and a kriging spatial predictor. Iranpanah *et al.* [10] used this method

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to analysis the finite strain data across a thrust sheet. In SBB method, first the locations are partitioned and then bootstrap algorithm is performed by resampling the separate blocks. Precision of the estimators in this bootstrap method is sensitive to block size selection. Specifying optimum block size for time series block bootstrap method has been studied by Kunsch [11], Hall *et al.* [7] and Lahiri [12]. Moreover, Nordman and Lahiri [15] proposed optimal block size for spatial subsampling. Nordman *et al.* [16] drive expressions for optimal block size for variance estimation by a spatial MBB method.

In this paper, we specify an optimum block size for SBB method in order to estimate asymptotically the variance of sample mean for spatial lattice data. Next, the asymptotic bias and variance of the sampled mean variance estimator are specified using separate block bootstrap method. Then, the optimum block size is obtained by minimizing the asymptotic mean square error of the estimator. Finally, the theoretical and asymptotic results are evaluated by a simulation study. In this case, the required preliminaries are presented in Section 3. Then we propose the separate block bootstrap method in Section 2. Section 4 consists of asymptotic determination of optimum block size and an empirical estimate for it. In Section 5, we discuss the optimum block size and its empirical estimation using Monte Carlo simulation of the spatial data. The last section will end with discussion and results.

Separate Block Bootstrap

Suppose the observations of a stationary random field; $\{Z(\boldsymbol{s}): \boldsymbol{s} \in \mathbb{Z}^d\}$ which is weakly dependent on the locations $\boldsymbol{S}_n \equiv \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_{N_n}\}$ inside the sampling region; $D_n \subset \mathbb{R}^d$ are presented as data set; $\boldsymbol{Z}_n = \{Z(\boldsymbol{s}): \boldsymbol{s} \in \boldsymbol{S}_n = D_n \cap \mathbb{Z}^d\}$. To consider asymptotic properties of bootstrap estimator, we assume that the sampling region D_n is unbounded as $n \to \infty$. This structure was used to study the asymptotic properties of spatial data as an increasing domain (Cressie [3]). Now assume D_0 is a Borel subset of $(-1/2, 1/2]^d$, consisting of an open neighborhood of the origin, so that for each positive sequence of real numbers $a_n \to \infty$, the number of cubes of the scaled lattice formed from $\overline{D_0} \cap \overline{D_0^c}$; i.e. $a_n \mathbb{Z}^d$, is of order of $O\left(\left(a_n^{-1}\right)^{d-1}\right)$, where $\overline{D_0}$ and $\overline{D_0^c}$ are closures of D_0 and D_0^c, respectively. Then, assume that $\{\lambda_n\}_{n\geq 1}$ is a sequence of real numbers not less than 1, such that $\lambda_n \to \infty$ as $n \to \infty$. Now we consider the sampling region as

$$D_n = \lambda_n^d D_0 \tag{1}$$

which is defined by inflating the prototype set D_0 by the scaling factor λ_n. In this case, volume of the sampling region is given by $|D_n| = \lambda_n^d |D_0|$ which is related to the sample volume by $N_n = \left|D_n \cap \mathbb{Z}^d\right|$, where $|A|$ is the cardinality of a countable set; $A \subset \mathbb{Z}^d$ or the lebesque measure of an uncountable set; $A \subset \mathbb{R}^d$. The structure of the above mentioned sampling region is similar to the MBB method (Lahiri [13]) and spatial subsampling (Nordman and Lahiri [15]). If $\hat{\theta}_n = t_n(\boldsymbol{Z}_n)$ is an estimator of the parameter θ based on $\boldsymbol{Z}_n$ observations, then the goal is to estimate the variance of the normalized statistic; $\sqrt{N_n}\hat{\theta}_n$ i.e. $\sigma_n^2 = N_n \mathrm{Var}\left(\hat{\theta}_n\right)$ using SBB method.

To conduct the SBB method, the sampling region D_n must be partitioned into cubic blocks. Assume that $\{\beta_n\}_{n\geq 1}$ is a sequence of positive integers so that $\beta_n^{-1} + \beta_n \lambda_n^{-1} = o(1)$, as $n \to \infty$. This means that β_n, called as block size, tends to infinity more slowly than the scaling factor λ_n in (1). Assume that $K_n = \left\{\boldsymbol{k} \in \mathbb{Z}^d : \beta_n(\boldsymbol{k} + U) \subset D_n\right\}$ is a set of separate, equal and complete d-dimensional cubic blocks indexed in the form of $\beta_n(\boldsymbol{k} + U)$ which are in the sampling region D_n, where $U = (0,1]^d$ is the unit cube in $\mathbb{R}^d$. Assume that $\boldsymbol{Z}_n(D_n) = \left\{Z(s_1), \ldots, Z\left(s_{N_n}\right)\right\}$ is a complete sample and $\boldsymbol{Z}_n(D_n(\boldsymbol{k}))$ is the subsample inside the **k**-th block, i.e.

$$D_n(\boldsymbol{k}) \equiv \beta_n(\boldsymbol{k} + U) \cap D_n, \quad \boldsymbol{k} \in K_n. \tag{2}$$

Regarding the new structure of the sampling region on the basis of blocks, the new sample volume is $N_{1n} = |K_n| B_n \leq N_n$, where $B_n = \left|\beta_n U \cap \mathbb{Z}^d\right| = \beta_n^d$ is the volume of each block. For simplicity, we will assume that $N_n = N_{1n}$, i.e. the sampling region D_n is covered by $|K_n|$ blocks of volume β_n. In order to achieve a spatial separate block bootstrap sample, firstly, a block is randomly selected for each $\boldsymbol{k} \in K_n$ from the set of separate blocks $\{D_n(\boldsymbol{k}):\boldsymbol{k}\in K_n\}$ and independent from other blocks. Then, using the observations in all $\boldsymbol{k}$ resampled blocks, and by joining them together, the bootstrap sample is obtained. In other words, assuming that $\{I_{\boldsymbol{k}}:\boldsymbol{k}\in K_n\}$ is a set of i.i.d. random variables with common distribution

$$P(I_k=\boldsymbol{i})=\frac{1}{|K_n|},\quad \boldsymbol{i}\in K_n, \tag{3}$$

then for each $\boldsymbol{k}\in K_n$, the subsample of separate block bootstrap is achieved as $\boldsymbol{Z}_n^*(D_n(\boldsymbol{k}))=\boldsymbol{Z}_n(D_n(I_{\boldsymbol{k}}))$. Now, the separate block bootstrap sample; $\boldsymbol{Z}_n^*(D_n)$ is specified through joining up the observations in the resampled blocks as $\{\boldsymbol{Z}_n^*(D_n(\boldsymbol{k})):\boldsymbol{k}\in K_n\}$. Then, the separate block bootstrap estimator of $\hat{\theta}_n$ and σ_n^2 are defined as $\hat{\theta}_n^*=t_n(\boldsymbol{Z}_n^*(D_n))$ and $\hat{\sigma}_n^2(\beta_n)=N_n\mathrm{Var}_*(\hat{\theta}_n^*)$ respectively, where Var_* denotes the bootstrap conditional variance given $\boldsymbol{Z}_n$ observations. When $\hat{\sigma}_n^2(\beta_n)$ does not have a closed form, a Monte Carlo simulation and B times repetition of the previous processes gives $\hat{\theta}_{n,1}^*,\ldots,\hat{\theta}_{n,B}^*$, then the separate block bootstrap estimate of $\hat{\sigma}_n^2(\beta_n)$ is approximated by

$$\hat{\sigma}_n^2(\beta_n)\simeq N_n\widehat{\mathrm{Var}_*}(\hat{\theta}_n^*)=\frac{N_n}{B}\sum_{b=1}^{B}\left(\hat{\theta}_{n,b}^*-\frac{1}{B}\sum_{b=1}^{B}\hat{\theta}_{n,b}^*\right)^2.$$

When the bootstrap estimation for the sample mean $\hat{\theta}_n=\bar{Z}_n$ is required, Iranpanah and Mohammadzadeh [8] showed that $\hat{\sigma}_n^2(\beta_n)$ is a consistent estimator for

$$\begin{aligned}\sigma_\infty^2&=\lim_{n\to\infty}N_n\mathrm{Var}(\bar{Z}_n)\\&=\sum_{\mathrm{k}\in\mathbb{Z}^d}\mathrm{E}[Z(0)-\mu][Z(\mathrm{k})-\mu].\end{aligned}$$

Iranpanah and Mohammadzadeh [9] also proved a similar property for kriging spatial predictor. However since the precision of the estimator $\hat{\sigma}_n^2(\beta_n)$ is severely sensitive to the block size β_n, finding the optimum block size will be considered in the following section.

Preliminaries

Assume that under sampling structure of the previous section, the sampled mean $\bar{Z}_n=N_n^{-1}\sum_{i=1}^{N_n}Z(s_i)$ is an estimator for the mean of random field, i.e. $\mu=\mathrm{E}[Z(\cdot)]$, based on $\boldsymbol{Z}_n$ observations. If $\bar{Z}_n^*=N_n^{-1}\sum_{i=1}^{N_n}Z^*(s_i)$ is the sample mean based on the bootstrap sample $\boldsymbol{Z}_n^*$ in SBB method, then the bootstrap estimate of $\sigma_n^2=N_n\mathrm{Var}(\bar{Z}_n)$ will be considered as $\hat{\sigma}_n^2(\beta_n)=N_n\mathrm{Var}_*(\bar{Z}_n^*)$. To do so, first some preliminary definitions, assumptions, and conditions that are required in the next lemmas and theorems will be presented.

For the vector $\boldsymbol{x}=(x_1,\ldots,x_d)'\in\mathbb{R}^d$, the Euclidean, L^1 and L^∞ norms are represented as $\|\boldsymbol{x}\|=\sum_{i=1}^d x_i^2$, $\|\boldsymbol{x}\|_1=\sum_{i=1}^d|x_i|$ and $\|\boldsymbol{x}\|_\infty=\max_{1\le i\le d}\{|x_i|\}$ respectively. Distance of the two sets $E_1,E_2\subset\mathbb{R}^d$ is defined as $\mathrm{dis}(E_1,E_2)=\inf\{\|\boldsymbol{x}-\boldsymbol{y}\|_\infty:\boldsymbol{x}\in E_1,\boldsymbol{y}\in E_2\}$.

Assume that $F_Z(T)$ denotes the σ-field generated by random variables; $\{Z(s):s\in T\subset\mathbb{Z}^d\}$. If $\tilde{\alpha}(T_1,T_2)=\sup\{|P(A\cap B)-P(A)P(B)|:A\in F_Z(T_1),B\in F_Z(T_2)\}$ where $T_1,T_2\subset\mathbb{Z}^d$ then the $\alpha-$mixing index for the random field is defined as

$$\begin{aligned}&\alpha(k,\ell)\\&=\sup\{\tilde{\alpha}(T_1,T_2):T_i\subset\mathbb{Z}^d,|T_i|\le\ell,\\&\quad i=1,2;\mathrm{dis}(T_1,T_2)\ge k\}.\end{aligned}\tag{4}$$

The required assumptions and conditions for the following lemmas and theorems are:

(i) If $n\to\infty$ then $\beta_n^{-1}+\beta_n^{(d+1)/d}\lambda_n^{-1}=o(1)$.

(ii) $\sigma_\infty^2=\sum_{\boldsymbol{k}\in\mathbb{Z}^d}\sigma(\boldsymbol{k})\in(0,\infty)$, where $\sigma(\boldsymbol{k})=\mathrm{Cov}(Z(\boldsymbol{s}),Z(\boldsymbol{s}+\boldsymbol{k}))$.

(iii) $\sup\{\tilde{\alpha}(T_1,T_2):T_1,T_2\subset\mathbb{Z}^d,|T_1|=l,\mathrm{dis}(T_1,T_2)\ge k\}=o(k^{-d})$.

(iv) There exist non-negative functions $\alpha_1(\cdot)$ and $g(\cdot)$, so that $\lim_{k\to\infty}\alpha_1(k)=0$, $\lim_{\ell\to\infty}g(\ell)=\infty$ and $\alpha(k,\ell)\le\alpha_1(k)g(\ell),k>0,\ell>0$.

($\mathbf{v_r}$) For $r\in\mathbb{Z}^+$, $1<\delta\le1$, $0<p<(2r-1-1/d)(2r+\delta)/\delta$, $x\in[1,\infty)$ and $c>0$, we have

$$\mathrm{E}|Z(s)|^{2r+\delta}<\infty,\ \sum_{m=1}^{\infty}m^{(2r-1)d-1}\alpha_1(m)^{\delta/(2r+\delta)}<\infty$$

and $g(x)\le cx^p$.

The growth rates for the blocks and the sampling region; D_n are presented in the assumption (i). The assumption (ii) shows that finite asymptotic variance $\sigma_\infty^2 = \lim_{n\to\infty} \sigma_n^2$ exists. The central limit theorem (Bolthausen [1]) is valid for $Z(\cdot)$ on the sets of increasing domain under the assumption (iii), having limits on D_0, assumption (iv) and conditions ($\mathrm{v_r}$). The assumption (iv) is a proper bound for $\alpha-$mixing index in the equation (4). The assumption (iv) and conditions ($\mathrm{v_r}$) also provide proper bounds for moments of observations. For the random fields under the assumption (iv), the distance bound $\alpha_1(\cdot)$ decreases with an exponential rate while the size of $g(\cdot)$ increases with a polynomial rate. The assumptions (ii)-(iv) are needed for mixing and momentum conditions presented in the conditions ($\mathrm{v_r}$). Some examples of random fields with weak dependency under the assumption (iv) and conditions ($\mathrm{v_r}$) are: Gaussian random fields with analytical spectral density, certain linear fields with moving average representation or autoregressive such as m-dependent fields, $AR(1)\times AR(1)$ separable lattice processes for modeling in $\mathbb{R}^2$, some Markov and Gibbs random fields and time series models (Doukhan [4]).

Optimal Block Size

In this section, without loss of generality, we assume $\mu = 0$.

Lemma 1. In SBB method, if $\bar{Z}_{\boldsymbol{k},n}$ is the sample mean of B_n observations in the $\boldsymbol{k}$ th block ($\boldsymbol{k}\in K_n$), then $\mathrm{E}_*\left(\bar{Z}_n^*\right) = \bar{Z}_n$ and $\mathrm{Var}_*\left(\bar{Z}_n^*\right) = |K_n|^2 \sum_{\boldsymbol{k}\in K_n}\left(\bar{Z}_{\boldsymbol{k},n} - \bar{Z}_n\right)^2$.

Proof. Since, each $\boldsymbol{k}\in K_n$ blocks in SBB method is achieved as i.i.d. through common distribution in (3), then

$$\begin{aligned}\mathrm{E}_*\left(\bar{Z}_n^*\right) &= \mathrm{E}_*\left[N_n^{-1}\sum_{i=1}^{N_n} Z^*(s_i)\right]\\ &= \mathrm{E}_*\left[|K_n|^{-1}\sum_{\boldsymbol{k}\in K_n}\bar{Z}^*_{\boldsymbol{k},n}\right]\\ &= \mathrm{E}_*\left(\bar{Z}^*_{\boldsymbol{0},n}\right)\\ &= |K_n|^{-1}\sum_{\boldsymbol{k}\in K_n}\bar{Z}_{\boldsymbol{k},n}\\ &= N_n^{-1}\sum_{i=1}^{N_n} Z(s_i)\\ &= \bar{Z}_n,\end{aligned}$$

$$\begin{aligned}\mathrm{Var}_*\left(\bar{Z}_n^*\right) &= \mathrm{Var}_*\left[N_n^{-1}\sum_{i=1}^{N_n} Z^*(s_i)\right]\\ &= \mathrm{Var}_*\left[|K_n|^{-1}\sum_{\boldsymbol{k}\in K_n}\bar{Z}^*_{\boldsymbol{k},n}\right]\\ &= |K_n|^{-1}\,\mathrm{Var}_*\left(\bar{Z}^*_{\boldsymbol{0},n}\right)\\ &= |K_n|^{-2}\sum_{\boldsymbol{k}\in K_n}\left(\bar{Z}_{\boldsymbol{k},n} - \bar{Z}_n\right)^2. \quad \square\end{aligned}$$

Lemma 2. (Doukhan [4]): Assume T_1 and T_2 as subsets of $\mathbb{Z}^d$ and p and q are non-negative values that satisfy in $1/p + 1/q < 1$. If the random variables; X_i are measurable with respect to $F_Z(T_i), i = 1,2$, then

$$\begin{aligned}&\left|\mathrm{Cov}(X_1, X_2)\right|\\ &\quad\le 8\left(\mathrm{E}|X_1|^p\right)^{1/p}\left(\mathrm{E}|X_2|^q\right)^{1/q}\\ &\qquad\alpha\left[\mathrm{dis}(T_1,T_2);\max\left(|T_1|,|T_2|\right)\right]^{1-1/p+1/q},\end{aligned}$$

where expectations exist and $\mathrm{dis}(T_1,T_2) > 0$.

Lemma 3. (Doukhan [4]): If $r\in\mathbb{Z}^+$, then under the conditions (iii)-($\mathrm{v_r}$) for $1\le m\le 2r$ and each $T\subset\mathbb{Z}^d$, we have

$$\mathrm{E}\left|\bar{Z}_n\right|^m \le C(\alpha) N_n^{-m/2}, \mathrm{E}\left|\sum_{\boldsymbol{s}\in T} Z(\boldsymbol{s})\right|^m \le C(\alpha)|T|^{-m/2},$$

where $C(\alpha)$ is a fixed value that depends only on the coefficient $\alpha(k,\ell), \ell\le 2r$ and $\mathrm{E}\left|Z(\boldsymbol{s})\right|^{2r+\delta}$.

Lemma 4. (Bolthausen [2]): Under the conditions (ii)-($\mathrm{v_r}$), $\sqrt{B_n}\bar{Z}_{0,n} \to_d Z_\infty$ as $n\to\infty$, and for $j = 1,2$, we have $B_n^j \mathrm{E}\left(\bar{Z}_{0,n}\right)^{2j} \to \mathrm{E}\left(Z_\infty\right)^{2j} = (2j-1)\sigma_\infty^{2j}$, where $\bar{Z}_{0,n}$ is the sample mean of the block including the origin, and $Z_\infty \sim N\left(0,\sigma_\infty^2\right)$.

Lemma 5. Under the assumption (i), $N_n/\lambda_n^d|D_0| \to 1$ as $n\to\infty$.

Proof. It is enough to show that $\left|N_n - \lambda_n^d|D_0|\right| \le C\lambda_n^{d-1}$. Therefore, first, we have to find an upper bound for N_n.

$$N_n \le \left|\left\{\boldsymbol{i}\in\mathbb{Z}^d : \left(\boldsymbol{i} + (-1/2,1/2]^d\right)\cap\lambda_n\overline{D_0^c} = \varnothing\right\}\right|$$

$$
\begin{aligned}
&+\left|\left\{ \boldsymbol{i}\in\mathbb{Z}^d : T^{\boldsymbol{i}}\cap\lambda_n\overline{D_0^c}\neq\varnothing,\right.\right.\\
&\qquad\left.\left. T^{\boldsymbol{i}}\cap\lambda_n\overline{D_0}\neq\varnothing; T^{\boldsymbol{i}}=\boldsymbol{i}+\left(-1/2,1/2\right]^d\right\}\right|\\
&\le\lambda_n\left|D_0\right|+\left|\left\{ \boldsymbol{i}\in\lambda_n^{-1}\mathbb{Z}^d : T^{i}\cap\lambda_n\overline{D_0^c}\neq\varnothing,\right.\right.\\
&\qquad\left.\left. T^{i}\cap\lambda_n\overline{D_0}\neq\varnothing; T^{i}=i+\left[-1,1\right]^d\right\}\right|\\
&\le\lambda_n\left|D_0\right|+2^d\left|\left\{ \boldsymbol{i}\in\lambda_n^{-1}\mathbb{Z}^d : T^{i}\cap\lambda_n\overline{D_0^c}\neq\varnothing,\right.\right.\\
&\qquad\left.\left. T^{i}\cap\lambda_n\overline{D_0}\neq\varnothing; T^{i}=i+\lambda_n^{-1}\left[0,1\right]^d\right\}\right|\\
&\le\lambda_n\left|D_0\right|+C\lambda_n^{d-1}.
\end{aligned}
$$

The last inequality is obtained from the bound condition on D_0. In the same way a lower bound for N_n, is specified as

$$
\begin{aligned}
N_n&\ge\left|\left\{ \boldsymbol{i}\in\mathbb{Z}^d : \left(\boldsymbol{i}+\left(-1/2,1/2\right]^d\right)\cap\lambda_n\overline{D_0^c}\neq\varnothing\right\}\right|\\
&\ge\left|\left\{ \boldsymbol{i}\in\mathbb{Z}^d : \left(\boldsymbol{i}+\left(-1/2,1/2\right]^d\right)\cap\lambda_n\overline{D_0^c}=\varnothing\right\}\right|\\
&\quad-\left|\left\{ \boldsymbol{i}\in\mathbb{Z}^d : T^{i}\cap\lambda_n\overline{D_0^c}\neq\varnothing, T^{i}\cap\lambda_n\overline{D_0}\neq\varnothing;\right.\right.\\
&\qquad\left.\left. T^{i}=\boldsymbol{i}+\left(-1/2,1/2\right]^d\right\}\right\}\\
&\ge\lambda_n\left|D_0\right|-C\lambda_n^{d-1}. \quad\square
\end{aligned}
$$

Lemma 6. Under the assumptions and conditions (ii)-($\mathrm{v_r}$), $\sigma_n^2-\sigma_\infty^2=O\left(N_n^{-1/d}\right)$, as $n\to\infty$.

Proof. For each $\boldsymbol{k}\in\mathbb{Z}^d$ assume that $N_n\left(\boldsymbol{k}\right)\ge\left|\left\{\boldsymbol{i}\in D_n\cap\mathbb{Z}^d : \boldsymbol{i}+\boldsymbol{k}\in D_n\right\}\right|$ to be the number of common locations in the sampling region of D_n and its $\boldsymbol{k}$-transfer. It is obvious that $N_n\left(\boldsymbol{k}\right)\le N_n$ and therefore taking into account the bound condition on D_0, we have

$$
\begin{aligned}
N_n&\le N_n\left(\boldsymbol{k}\right)+\left|\left\{ \boldsymbol{i}\in\mathbb{Z}^d : T^{i}\cap\lambda_n\overline{D_0^c}\neq\varnothing,\right.\right.\\
&\qquad\left.\left. T^{i}\cap\lambda_n\overline{D_0}\neq\varnothing; T^{i}=\boldsymbol{i}+\left\|\boldsymbol{k}\right\|_\infty\left[-1,1\right]^d\right\}\right|\\
&\le N_n\left(\boldsymbol{k}\right)+\left(3\left\|\boldsymbol{k}\right\|_\infty\right)^d\left|\left\{ \boldsymbol{i}\in\lambda_n^{-1}\mathbb{Z}^d : T^{i}\cap\lambda_n\overline{D_0^c}\neq\varnothing,\right.\right.\\
&\qquad\left.\left. T^{i}\cap\lambda_n\overline{D_0}\neq\varnothing; T^{i}=\boldsymbol{i}+\lambda_n^{-1}\left[0,1\right]^d\right\}\right|\\
&\le N_n\left(\boldsymbol{k}\right)+C3\left\|\boldsymbol{k}\right\|_\infty^d\lambda_n^{d-1}.
\end{aligned}
\tag{5}
$$

Also using Lemma 2 and stationarity of $Z\left(\cdot\right)$, for each $0\neq\boldsymbol{k}\in\mathbb{Z}^d$, we have

$$
\begin{aligned}
\left|\sigma\left(\boldsymbol{k}\right)\right|&\le 8\left[\mathrm{E}\left|Z\left(\boldsymbol{s}\right)\right|^{(2r+\delta)/r}\right]^{2r/(2r+\delta)}\alpha\left(\left\|\boldsymbol{k}\right\|_\infty,1\right)^{\delta/(2r+\delta)}\\
&\le C\alpha_1\left(\left\|\boldsymbol{k}\right\|_\infty\right)^{\delta/(2r+\delta)}.
\end{aligned}
\tag{6}
$$

Since $\left|\left\{\boldsymbol{k}\in\mathbb{Z}^d : \left\|\boldsymbol{k}\right\|_\infty=m\right\}\right|\le 4\left(2m+1\right)^{d-1}$, tacking summation of both sides of (6) implies that the covariances are absolutely sumable on $\mathbb{Z}^d$, i.e.

$$
\sum_{\boldsymbol{k}\in\mathbb{Z}^d}\left|\sigma\left(\boldsymbol{k}\right)\right|\le\sigma\left(\boldsymbol{0}\right)+C\sum_{m=1}^{\infty}2\left(2m+1\right)^{d-1}\alpha_1\left(m\right)^{\delta/(2r+\delta)}<\infty. \tag{7}
$$

Finally, by equations (5) and (7) and Lemma 5, we have

$$
\begin{aligned}
\sigma_n^2-\sigma_\infty^2&=N_n^{-1}\mathrm{Var}\left[\sum_{\boldsymbol{s}\in D_n\cap\mathbb{Z}^d}Z\left(\boldsymbol{s}\right)\right]-\sigma_\infty^2\\
&=N_n^{-1}\sum_{\boldsymbol{k}\in\mathbb{Z}^d}N_n\left(\boldsymbol{k}\right)\sigma\left(\boldsymbol{k}\right)-\sigma_\infty^2\\
&\le N_n^{-1}\sum_{\boldsymbol{k}\in\mathbb{Z}^d}\left|N_n-N_n\left(\boldsymbol{k}\right)\right|\sigma\left(\boldsymbol{k}\right)\\
&\le CN_n^{-1}\lambda_n^{d-1}\sum_{m=1}^{\infty}m^{2d-1}\alpha_1\left(m\right)^{\delta/(2r+\delta)}\\
&=O\left(N_n^{-1/d}\right). \quad\square
\end{aligned}
$$

To specify optimum block size β_n through minimizing asymptotic $\mathrm{MSE}\left[\hat{\sigma}_n^2\left(\beta_n\right)\right]$, we must specify asymptotic bias and variance of the separate block bootstrap estimator:

$$
\hat{\sigma}_n^2\left(\beta_n\right)=B_n\left|K_n\right|^{-1}\sum_{\boldsymbol{k}\in K_n}\left(\bar{Z}_{\boldsymbol{k},n}-\bar{Z}_n\right)^2. \tag{8}
$$

Theorem 1. Under the assumptions and conditions (i)-($\mathrm{v_r}$), the asymptotic bias of $\hat{\sigma}_n^2\left(\beta_n\right)$ equals to

$$
\mathrm{E}\left[\hat{\sigma}_n^2\left(\beta_n\right)\right]-\sigma_n^2=-\frac{B_0}{\beta_n}\left(1+o\left(1\right)\right); B_0=\sum_{\boldsymbol{k}\in\mathbb{Z}^d}\left\|\boldsymbol{k}\right\|_1\sigma\left(\boldsymbol{k}\right).
$$

Proof. For each $\boldsymbol{k}\in\mathbb{Z}^d$, suppose that $B_n\left(\boldsymbol{k}\right)=\left|\left\{\boldsymbol{i}\in D_n\left(\boldsymbol{0}\right)\cap\mathbb{Z}^d : \boldsymbol{i}+\boldsymbol{k}\in D_n\left(\boldsymbol{0}\right)\cap\mathbb{Z}^d\right\}\right|$ is the number of common block locations including the origin $D_n\left(\boldsymbol{0}\right)$ and its $\boldsymbol{k}-$transfer. It is clear that $B_n\left(\boldsymbol{k}\right)\le B_n$ taking into account the equation (8), stationarity of the process, the assumption 1 and the Lemmas 3 and 5, we have

$$\begin{aligned}
\mathrm{E}\left[\hat{\sigma}_n^2(\beta_n)\right] &= B_n \mathrm{E}\left(|K_n|^{-1}\sum_{\boldsymbol{k}\in K_n}\bar{Z}_{\boldsymbol{k},n}^2 - \bar{Z}_n^2\right)\\
&= B_n\left[\mathrm{E}\left(\bar{Z}_{\boldsymbol{0},n}^2\right)-\mathrm{E}\left(\bar{Z}_n^2\right)\right]\\
&= B_n^{-1}\mathrm{E}\left[\sum_{\boldsymbol{s}\in D_n(\boldsymbol{0})\cap\mathbb{Z}^d} Z(\boldsymbol{s})\right]^2 + O\left(B_n/N_n\right) \qquad (9)\\
&= B_n^{-1}\sum_{\boldsymbol{k}\in\mathbb{Z}^d} B_n(\boldsymbol{k})\sigma(\boldsymbol{k}) + o\left(\beta_n^{-1}\right)\\
&= -\frac{\beta_n^{d-1}}{B_n}\sum_{\boldsymbol{k}\in\mathbb{Z}^d}\frac{B_n - B_n(\boldsymbol{k})}{\beta_n^{d-1}}\sigma(\boldsymbol{k}) + \sigma_\infty^2 + o\left(\beta_n^{-1}\right).
\end{aligned}$$

For each $\boldsymbol{k}\in\mathbb{Z}^d$, the block containing the origin $D_n(\mathbf{0})$, as in Lemmas 5 and 6, for the sampling region D_n, we can show that

$$\begin{aligned}
0 &\le B_n - B_n(\boldsymbol{k})\\
&\le C\,\|\boldsymbol{k}\|_\infty^d\,\beta_n^{d-1}, \beta_n^{-(d-1)}\left(B_n - B_n(\boldsymbol{k})\right)\to\|\boldsymbol{k}\|_1 .
\end{aligned} \qquad (10)$$

As a result, by the equations (6) and (7) in Lemma 6, we have

$$\begin{aligned}
\sum_{\boldsymbol{k}\in\mathbb{Z}^d}\left|\frac{B_n - B_n(\boldsymbol{k})}{\beta_n^{d-1}}\sigma(\boldsymbol{k})\right| &\le C\sum_{\boldsymbol{k}\in\mathbb{Z}^d}\|\boldsymbol{k}\|_\infty^d\left|\sigma(\boldsymbol{k})\right|\\
&\le C\sum_{m=1}^{\infty} m^{2d-1}\alpha_1(m)^{\delta/(2r+\delta)} \qquad (11)\\
&<\infty.
\end{aligned}$$

Now, using the dominated lebesque convergence theorem and equation (10), we have

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^d}\frac{B_n - B_n(\boldsymbol{k})}{\beta_n^{d-1}}\sigma(\boldsymbol{k}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d}\|\boldsymbol{k}\|_1\,\sigma(\boldsymbol{k})\left(1+o(1)\right). \qquad (12)$$

Then, using assumption (i) and the Lemmas 5 and 6, $\sigma_n^2-\sigma_\infty^2 = o\left(\beta_n^{-1}\right)$. Therefore, on the basis of the equations (9) and (12), we have

$$\begin{aligned}
\mathrm{E}\left[\hat{\sigma}_n^2(\beta_n)\right]-\sigma_n^2 &= -\frac{1}{\beta_n}\sum_{\boldsymbol{k}\in\mathbb{Z}^d}\|\boldsymbol{k}\|_1\sigma(\boldsymbol{k})\left(1+o(1)\right)\\
&= -\frac{B_0}{\beta_n}\left(1+o(1)\right). \quad \square
\end{aligned}$$

Theorem 2. Under the assumptions and conditions (i)-($\mathrm{v_r}$), we have

$$\mathrm{Var}\left[\hat{\sigma}_n^2(\beta_n)\right] = \frac{2\sigma_\infty^2 B_n}{N_n}\left(1+o(1)\right).$$

Proof. Taking into account equation (8), we have

$$\begin{aligned}
\mathrm{Var}\left[\hat{\sigma}_n^2(\beta_n)\right] &= B_n^2\mathrm{Var}\left(|K_n|^{-1}\sum_{\boldsymbol{k}\in K_n}\bar{Z}_{\boldsymbol{k},n}^2 - \bar{Z}_n^2\right)\\
&= B_n^2\Bigg[|K_n|^{-2}\,\mathrm{Var}\left(\sum_{\boldsymbol{k}\in K_n}\bar{Z}_{\boldsymbol{k},n}^2\right) \qquad (13)\\
&\qquad + \mathrm{Var}\left(\bar{Z}_n^2\right) - 2|K_n|^{-1}\sum_{\boldsymbol{k}\in K_n}\mathrm{Cov}\left(\bar{Z}_{\boldsymbol{k},n}^2,\bar{Z}_n^2\right)\Bigg]\\
&= T_1 + T_2 - 2T_3.
\end{aligned}$$

Since the process is stationary, we have

$$\begin{aligned}
T_1 &= B_n^2|K_n|^{-2}\,\mathrm{Var}\left(\sum_{\boldsymbol{k}\in K_n}\bar{Z}_{\boldsymbol{k},n}^2\right)\\
&= B_n^2|K_n|^{-2}\Bigg[\sum_{\boldsymbol{k}\in K_n}\mathrm{Var}\left(\bar{Z}_{\boldsymbol{k},n}^2\right)\\
&\qquad + \sum_{\boldsymbol{k}\in K_n}\sum_{\boldsymbol{l}(\ne\boldsymbol{k})\in K_n}\mathrm{Cov}\left(\bar{Z}_{\boldsymbol{k},n}^2,\bar{Z}_{\boldsymbol{l},n}^2\right)\Bigg] \qquad (14)\\
&= B_n^2|K_n|^{-1}\Bigg[\mathrm{Var}\left(\bar{Z}_{\boldsymbol{0},n}^2\right)\\
&\qquad + \sum_{\boldsymbol{0}\ne\boldsymbol{k}\in K_n}\mathrm{Cov}\left(\bar{Z}_{\boldsymbol{0},n}^2,\bar{Z}_{\boldsymbol{k},n}^2\right)\Bigg]\\
&= T_4 + T_5.
\end{aligned}$$

With regard to the Lemma 3, $\mathrm{E}\left(B_n\bar{Z}_{\boldsymbol{0},n}^2\right)\to\mathrm{E}\left(Z_\infty^2\right)=\sigma_\infty^2$ and $\mathrm{E}\left(B_n^2\bar{Z}_{\boldsymbol{0},n}^4\right)\to\mathrm{E}\left(Z_\infty^4\right)$ $=3\sigma_\infty^4$, as $n\to\infty$. Therefore $\mathrm{Var}\left(B_n\bar{Z}_{\boldsymbol{0},n}^2\right)\to$ $\mathrm{Var}\left(Z_\infty^2\right)=2\sigma_\infty^4$. As a result, we have

$$T_4 = |K_n|^{-1}\,\mathrm{Var}\left(Z_\infty^2\right)\left(1+o(1)\right) = 2\sigma_\infty^4 B_n N_n^{-1}\left(1+o(1)\right). \qquad (15)$$

For each $\boldsymbol{k}\in K_n$, we assume that $\mathrm{dis_n}(\boldsymbol{k})\equiv\mathrm{dis}\left[D_n(\boldsymbol{0})\cap\mathbb{Z}^d, D_n(\boldsymbol{k})\cap\mathbb{Z}^d\right]\ge\beta_n$ is the distance of the block including the origin $D_n(\mathbf{0})$ and separate block $D_n(\boldsymbol{k})$. As a result, due to the stationarity of the process, assumptions and conditions (iv) and ($\mathrm{v_r}$) and the Lemmas 2 and 3, we have

$$\begin{aligned}
\left|\mathrm{Cov}\left(\bar{Z}_{\boldsymbol{0},n}^2,\bar{Z}_{\boldsymbol{k},n}^2\right)\right| &\le 8\left(\mathrm{E}\left|\bar{Z}_{\boldsymbol{0},n}^2\right|^{(2r+\delta)/r}\right)^{2r/(2r+\delta)}\\
&\qquad \alpha\left[\mathrm{dis_n}(\boldsymbol{k}), B_n\right]^{\delta/(2r+\delta)}\\
&\le CB_n^{-2}\alpha_1\left[\mathrm{dis_n}(\boldsymbol{k})\right]^{\delta/(2r+\delta)}\beta_n^{(2r-1)d-1}.
\end{aligned}$$

Since $\left|\left\{\boldsymbol{k} \in K_n : \mathrm{dis_n}\left(\boldsymbol{k}\right) = m\right\}\right| \le C\left(\beta_n + m\right)^{d-1}$, we conclude that

$$\begin{aligned} T_5 &= B_n^{-2}\left|K_n\right|^{-1} \sum_{\boldsymbol{0}\neq\boldsymbol{k}\in K_n} \mathrm{Cov}\left(\bar{Z}^2_{\boldsymbol{0},n}, \bar{Z}^2_{\boldsymbol{k},n}\right) \\ &\le C\left|K_n\right|^{-1} \sum_{m=\beta_n}^{\infty} m^{(2r-1)d-1}\alpha_1\left(m\right)^{\delta/(2r+\delta)} \\ &= o\left(B_n/N_n\right). \end{aligned} \tag{16}$$

Using Lemma 3, we will have

$$\begin{aligned} T_2 &= B_n^2 \mathrm{Var}\left(\bar{Z}_n^2\right) \le B_n^2 \mathrm{E}\left(\bar{Z}_n^4\right) \\ &= O\left(B_n^2/N_n^2\right) = o\left(B_n/N_n\right). \end{aligned} \tag{17}$$

Using Cauchy-Schwartz inequality and equations (13)-(17) we can write

$$\begin{aligned} T_3 &\le \sqrt{T_1T_2}\left\{\left[2\sigma_\infty^4\left(B_n/N_n\right) + o\left(B_n/N_n\right)\right]o\left(B_n/N_n\right)\right\}^{1/2} \\ &= o\left(B_n/N_n\right). \end{aligned} \tag{18}$$

Finally, using the equations (13)-(18), Theorem 2 is proved. □

Theorems 1 and 2 show that $\hat{\sigma}_n^2\left(\beta_n\right)$ is a MSE-consistent estimator, so it is also consistent for σ_n^2. Bias and variance of the estimator $\hat{\sigma}_n^2\left(\beta_n\right)$ depend on the block size β_n. Increase in the block size β_n leads to decrease of the bias and increase of the variance estimator β_n. The best block size β_n, can be found by minimizing a combination of bias and variance of the estimator $\hat{\sigma}_n^2\left(\beta_n\right)$.

Theorem 3. Under the assumptions and conditions (i)-($\mathrm{v_r}$), the size of asymptotic optimum block size for $\hat{\sigma}_n^2\left(\beta_n\right)$ is determined by

$$\beta_n^{\mathrm{opt}} = \left(\frac{N_n B_0^2}{d\sigma_\infty^4}\right)^{1/(d+2)}\left(1+o\left(1\right)\right).$$

Proof. Value of β_n^{opt} can be achieved by minimization of

$$\begin{aligned} \mathrm{MSE}\left[\hat{\sigma}_n^2\left(\beta_n\right)\right] &= \left[\mathrm{Bias}\left(\hat{\sigma}_n^2\left(\beta_n\right)\right)\right]^2 + \mathrm{Var}\left[\hat{\sigma}_n^2\left(\beta_n\right)\right] \\ &= \left(\frac{B_0^2}{\beta_n^2} + \frac{2\sigma_\infty^4\beta_n^d}{N_n}\right)\left(1+o\left(1\right)\right), \end{aligned}$$

with respect to β_n. □

The optimum block size β_n^{opt} depends on two unknown parameters B_0 and σ_∞^2. In this paper we used the nonparametric plug-in method suggested by Lahiri *et al.* [14] to estimate these parameters as well as $\hat{\beta}_n$. This method was originally presented for time series but we have extended it to spatial lattice data. Suppose that the primary block sizes $\beta_{n,1}$ and $\beta_{n,2}$ are sequences of positive integers, so that they satisfy the assumption (i). On the basis of the primary block size $\beta_{n,1}$, the part variance σ_∞^2 is estimated as $\sigma_\infty^2 = \hat{\sigma}_n^2\left(\beta_{n,1}\right)$. Also for the biased element B_0, based on two variance estimates of separate block bootstrap using $\beta_{n,2}$, we presented the estimator $\hat{B}_0 = 2\beta_{n,2}\left[\hat{\sigma}_n^2\left(2\beta_{n,2}\right) - \hat{\sigma}_n^2\left(\beta_{n,2}\right)\right]$. Therefore using the nonparametric plug-in method the optimum block size is estimated as $\hat{\beta}_n = \left(N_n\hat{B}_0^2/d\hat{\sigma}_\infty^4\right)^{1/(d+2)}$.

Theorem 4. Under the assumptions and conditions (i)-($\mathrm{v_r}$), $\dfrac{\hat{\beta}_n}{\beta_n^{\mathrm{opt}}} \xrightarrow{P} 1$, as $n \to \infty$.

Proof. Using Theorems 1 and 2, $\hat{B}_0$ and $\hat{\sigma}_\infty^2$ are MSE-consistent estimators of B_0 and σ_∞^2, respectively. Therefore, $\hat{\beta}_n$ is a consistent estimator of β_n. □

Nonparametric plug-in estimator $\hat{\beta}_n$, depends on two primary block size parameters $\beta_{n,1}$ and $\beta_{n,2}$. Using the Theorem 3, the optimum rate of the primary block size of $\beta_{n,1}$ to estimate the variance part of σ_∞^2 equals to $N_n^{1/(d+2)}$ and for $\beta_{n,2}$ as the bias part of B_0, equals to $N_n^{1/(d+4)}$. Therefore, their acceptable choices are $\beta_{n,i} = C_i N_n^{1/(d+2i)}; i = 1,2$. Our numerical studies show that the proper values for C_1 and C_2 in the interval [0.5,2] are those with 0.25 distances and therefore we suggest that $C_1 = \left\{0.5, 0.75\right\}$, $C_2 = 0.5$, respectively.

Simulation Study

In this section, first we determine β_n^{opt} and then $\hat{\beta}_n$ by nonparametric plug-in method and evaluate them by a Monte Carlo simulation study. Suppose that $\left\{Z\left(\boldsymbol{s}\right) : \boldsymbol{s} \in \mathbb{N}^2\right\}$ is a second order stationary Gaussian random field with zero mean and exponential covariogram defined by

$$\sigma(h;\gamma)=\begin{cases} c_0+c_1 & \|h\|=0 \\ c_1 e^{\frac{-\|h\|}{a}} & \|h\|\neq 0 \end{cases}$$

where $\gamma=(c_0,c_1,a)$ are nugget effect, partial sill and range, respectively. Regarding the two models with parameters $\gamma_1=(0.5,0.5,0.5)$ and $\gamma_2=(1,1,1)$, we can generate the samples in a square regular grid in three regions through Choleski decomposition (Cressie [3]) method against $D_0=(0,1]^2$ and λ_n=12, 24, 48. If $\hat{\theta}_n=\bar{Z}_n$ is the sample mean in the grids, separate block bootstrap estimator of $\sigma_n^2=N_n\operatorname{Var}(\hat{\theta}_n)$ is given by $\hat{\sigma}_n^2(\beta_n)=B_n|K_n|^{-1}\sum_{k\in K_n}(\bar{Z}_{k,n}-\bar{Z}_n)^2$, where $\bar{Z}_{k,n}$ is the sample mean of $B_n=\beta_n^2$ observations in the separate blocks:

$$D_n(\boldsymbol{k})=(\beta_n k_1-\beta_n,\beta_n k_1]\times(\beta_n k_2-\beta_n,\beta_n k_2];\boldsymbol{k}=(k_1,k_2)'\in\mathbb{N}^2,0<k_1,k_2<\lambda_n\beta_n^{-1}.$$

We will consider separate block sizes β_n for the three values of λ_n, respectively (2, 3, 4, 6), (2, 3, 4, 6, 8, 12) and (2, 3, 4, 6, 8, 12, 16, 24). Then the value of σ_n^2 in model 1 for the three values of λ_n will be 1.430, 1.436, 1.480, and in model 2 it will be 6.311, 6.890, 7.193, respectively. The limit values of σ_∞^2 for the two models are 1.483 and 7.286, respectively. Now for the two models and the three values of λ_n and also the considered value of separate block size β_n, the amount of $\hat{\sigma}_n^2(\beta_n)$ are calculated.

Table 1 shows the approximate values of bias $\mathrm{E}\left[\hat{\sigma}_n^2(\beta_n)/\sigma_n^2-1\right]$, variance $\mathrm{E}\left[\left(\hat{\sigma}_n^2(\beta_n)-\mathrm{E}\hat{\sigma}_n^2(\beta_n)\right)/\sigma_n^2\right]^2$ and mean square error $\mathrm{E}\left[\hat{\sigma}_n^2(\beta_n)/\sigma_n^2-1\right]^2$, as relatively on the basis of 10000 repetitions of Monte Carlo simulation. As can be seen, an increase of the block size β_n leads to a decrease in the bias value and an increase in the variance value for both models and the three states of λ_n. These results are nearly in conformity to the asymptotic results in Theorems 1 and 2. The values of bias, variance and non relative MSE in model 2, with stronger correlation structure, are greater than the one in model 1, with weaker correlation structure and in conformity to the results of the Theorems 1, 2 and 3. Optimum block size values β_n^{opt} can be achieved through comparing MSE values and finding their minimum amounts which are for the three values of λ_n in the models 1 and 2 are 2, 3, 4 and 3, 6, 8, respectively. Comparison of the various values of β_n^{opt} shows that when the sample size $N_n=\lambda_n^2$ increases, the β_n^{opt} increases too. Also β_n^{opt} value is greater in the models with stronger correlation structure comparing to the models with weaker correlation structure.

Now the nonparametric plug-in estimator $\hat{\beta}_n=\left(N_n\hat{B}_0^2/d\hat{\sigma}_\infty^4\right)^{1/(d+2)}$ is evaluated numerically through a simulation study. Estimates of two quantities $\hat{B}_0=2\beta_{n,2}\left[\hat{\sigma}_n^2(2\beta_{n,2})-\hat{\sigma}_n^2(\beta_{n,2})\right]$ and $\hat{\sigma}_\infty^2=\hat{\sigma}_n^2(\beta_{n,1})$ depend on the primary block sizes $\beta_{n,1}$ and $\beta_{n,2}$ whose suggested values are $\beta_{n,i}=C_iN_n^{1/(d+2i)};i=1,2$. Also regarding the empirical results, in the present paper we used the values $C_1=\{0.5,0.75\}$ and $C_2=0.5$. Table 2 shows frequency of various values of the nonparametric plug-in estimates of the block sizes $\hat{\beta}_n=1,2,\ldots,8,9^+$ in 1000 time repetition of Monte Carlo simulation for the two models 1 and 2, the three values of λ_n and the two values of C_1. The last column of Table 2 shows optimum block size values of β_n^{opt} gained from Table 1 for various models. For example, for the first row of Table 2 at first two primary suggested block sizes $\beta_{n,1}=0.5(144)^{1/4}\simeq 2$ and $\beta_{n,2}=0.5(144)^{1/6}\simeq 2$ are calculated, then the separate block bootstrap estimate of $\hat{\sigma}_n^2(\beta_n)$ are obtained from blocks with the sizes 2 and 4 of the simulated data. Then two values $\hat{B}_0=4\left[\hat{\sigma}_n^2(4)-\hat{\sigma}_n^2(2)\right]$ and $\hat{\sigma}_\infty^2=\hat{\sigma}_n^2(2)$ are calculated and lastly the nonparametric plug-in estimate for the block size is obtained as $\hat{\beta}_n=\left(144\hat{B}_0^2/2\hat{\sigma}_\infty^4\right)^{1/4}$, which may be one of the block sizes $1,\cdots,9^+$ in Table 2. As can be seen β_n^{opt} equals the mode of $\hat{\beta}_n$ resulting from Table 1 in different situations. This shows that $\hat{\beta}_n$ is a proper estimate for β_n^{opt}.

Results and Discussion

Since precision of estimators in SBB method depends on block size, the optimum block size is asymptotically specified for bootstrap estimation of the sample mean variance of lattice data and its

Table 1. Approximations of bias, variance and MSE of the separate block bootstrap estimates of $\hat{\sigma}_n^2\left(\beta_n\right)$ as relatively on 10000 repetitions of Mont Carlo simulation

		Model 1			**Model 2**		
λ_n	β_n	Bias	Var	MSE	Bias	Var	MSE
12	2	-0.213	0.035	**0.081**	-0.555	0.013	0.322
	3	-0.184	0.087	0.121	-0.461	0.042	**0.254**
	4	-0.193	0.160	0.196	-0.406	0.090	0.255
	6	-0.290	0.335	0.419	-0.422	0.224	0.402
24	2	-0.211	0.009	0.053	-0.575	0.003	0.334
	3	-0.158	0.023	**0.047**	-0.463	0.010	0.224
	4	-0.132	0.044	0.061	-0.383	0.023	0.169
	6	-0.127	0.098	0.114	-0.297	0.069	**0.157**
	8	-0.163	0.176	0.203	-0.274	0.132	0.207
	12	-0.281	0.356	0.435	-0.344	0.281	0.400
48	2	-0.214	0.003	0.048	-0.588	0.001	0.347
	3	-0.155	0.007	0.031	-0.475	0.003	0.228
	4	-0.124	0.013	**0.029**	-0.389	0.006	0.158
	6	-0.091	0.033	0.041	-0.283	0.019	0.099
	8	-0.088	0.058	0.066	-0.227	0.039	**0.091**
	12	-0.090	0.133	0.141	-0.186	0.100	0.135
	16	-0.139	0.223	0.242	-0.196	0.185	0.224
	24	-0.264	0.409	0.478	-0.294	0.363	0.449

Table 2. Frequency $\hat{\beta}_n$ on 1000 repetitions of the Mont Carlo simulation

			$\hat{\beta}_n$									
Model	λ_n	C_1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	9^+	β_n^{opt}
	12	0.5	91	**292**	241	211	126	26	4	0	0	2
		0.75	126	**276**	213	188	118	50	14	3	2	
1	24	0.5	118	170	**275**	235	143	36	9	0	0	3
		0.75	100	205	**288**	256	121	14	5	1	0	
	48	0.5	53	131	250	**254**	211	84	7	0	0	4
		0.75	59	142	228	**253**	213	81	16	0	0	
	12	0.5	74	179	**216**	209	185	96	30	1	0	3
		0.75	101	186	**285**	212	146	51	13	1	0	
2	24	0.5	21	60	112	201	245	**294**	61	2	0	6
		0.75	7	10	46	102	296	**458**	81	0	0	
	48	0.5	0	0	4	10	50	197	286	**392**	61	8
		0.75	0	1	2	7	59	354	47	**530**	0	

nonparametric plug-in estimate has been presented. Also in a Monte Carlo simulation study on spatial data, the theoretical and asymptotic results have been evaluated. The optimum block size can be specified for kriging spatial predictor as in sample mean of spatial lattice data. Also, we can similarly consider the optimum block size for the bootstrap estimate of bias estimators of a random field.

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