Journal of Sciences, Islamic Republic of Iran 20(2): 159-166 (2009) http://jsciences.ut.ac.ir
University of Tehran, ISSN 1016-1104

Extension of Hardy Inequality on Weighted Sequence Spaces

R. Lashkaripour[1,*] and D. Foroutannia[2]

[1]*Department of Mathematics, Faculty of Mathematics, University of Sistan and Baluchestan, Zahedan, Islamic Republic of Iran*
[2]*Department of Mathematics, Faculty of Sciences, Vali-E-Asr University o f Rafsanjan, Rafsanjan, Islamic Republic of Iran*

Received: 25 June 2007 / Revised: 18 April 2009 / Accepted: 9 May 2009

Abstract

Let $p \geq 1$ and (w_n) be a sequence with non-negative entries. If $T = (t_{n,k}) \geq 0$, denote by $\|T\|_{p,w}$ the infimum of those U satisfying the following inequality:

$$\left(\sum_{n=1}^{\infty} w_n \left(\sum_{k=1}^{\infty} t_{n,k} a_k \right)^p \right)^{\frac{1}{p}} \leq U \left(\sum_{k=1}^{\infty} w_k a_k^p \right)^{\frac{1}{p}},$$

whenever $(a_n) \in l_p(w)$. The purpose of this paper is to give an upper bound for the norm of operator T on weighted sequence spaces $d(w,p)$ and $l_p(w)$ and also $e(w,\infty)$. We considered this problem for certain matrix operators such as Norlund, Weighted mean, Ceasaro and Copson matrices. This problem is considered by some authors like Bennett, Jamson and the first author on sequence spaces l_p and weighted sequence spaces for some kind of matrix operators. Also, this study is an extension of paper by Chang-Pao Chen, Dah-Chin Luor and Zong-Yin Ou.

Keywords: Hardy inequality; Norlund matrix; Weighted mean matrix

Introduction

In this study we consider the norm of certain matrix operators on weighted sequence spaces $l_p(w)$, $e(w,\infty)$ and Lorentz sequence spaces $d(w,p)$, $p \geq 1$, which is considered in [1] and [2] on l_p spaces and in [5-8] and [10] on $l_p(w)$ and $d(w,p)$ for some matrix operators such as Cesaro, Copson, Hausdorff and Hilbert operators.

Assume that l_p is the normed linear space of all sequences $a = (a_n)$ with finite norm $\|a\|_p$, where

$$\|a\|_p = \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p}.$$

Suppose that$w = (w_n)$ is a sequence with non-negative entries. For $p \geq 1$, we define the weighted sequence space $l_p(w)$ as follows:

2000 Mathematics Subject Classification. 26D15, 15A60.

* Corresponding author, Tel.: +98(541)2447166, Fax: +98(541)2447166, E-mail: lashkari@hamoon.usb.ac.ir

$$l_p(w)=\left\{(a_n):\sum_{n=1}^{\infty}w_n\left|a_n\right|^p<\infty\right\},$$

with norm, $\|.\|_{p,w}$, where

$$\|a\|_{p,w}=\left(\sum_{n=1}^{\infty}w_n\left|a_n\right|^p\right)^{1/p}.$$

Also, if (w_n) is a decreasing non-negative sequence such that $\lim_{n\to\infty}w_n=0$ and $\sum_{n=1}^{\infty}w_n=\infty$, then the Lorentz sequence space $d(w,p)$ is defined as follows:

$$d(w,p)=\left\{(a_n):\sum_{n=1}^{\infty}w_n a_n^{*p}<\infty\right\},$$

where (a_n^*) is the decreasing rearrangement of $(|a_n|)$. In fact, $d(w,p)$ is the space of null sequences a for which a^* is in $l_p(w)$, with norm $\|a\|_{d(w,p)}=\|a^*\|_{p,w}$.

Let $A_k^*=a_1^*+\cdots+a_k^*$ and $W_k=w_1+\cdots+w_k$, we define the weighted sequence space $e(w,\infty)$ as follows:

$$e(w,\infty)=\left\{(a_n):\sup_k\frac{A_k^*}{W_k}<\infty\right\},$$

with norm $\|.\|_{w,\infty}$, which is defined as follows:

$$\|a\|_{w,\infty}=\sup_k\frac{A_k^*}{W_k}.$$

Our objective in section 1 is to give a generalization of some results obtained by [1] and [2]. In section 2, we try to solve the problem of finding the norm of certain matrix operators on $d(w,1)$ and $e(w,\infty)$ and we deduce the existence of an upper bound for certain matrix operators such as Cesaro and Copson operators.

The problem of finding the lower bound of matrix operators on weighted sequence spaces is considered in [9].

Results

1. Matrix Operators on d(w, p) and $l_p(w)$

Now consider the operator $T=(t_{i,j})$ defined by $Ta=b$, where $b_i=\sum_{i=1}^{\infty}t_{i,j}a_j$. We write $\|T\|_{p,w}$ for the norm of T as an operator from $l_p(w)$ into itself, and $\|T\|_p$ for the norm of T as an operator from l_p into itself, and $\|T\|_{d(w,p)}$ for the norm of T as an operator on $d(w,p)$.

The following conditions is what we need to convert statements for $l_p(w)$ to ones for $d(w,p)$. We assume throughout that:

(1) For all i,j, $t_{i,j}\geq 0$.

(2) For all i, $\lim_{j\to\infty}t_{i,j}=0$.

(3) Either $t_{i,j}$ decreases with j for each i, or $t_{i,j}$ decreases with i for each j, and $c_{m,j}=\sum_{i=1}^{m}t_{i,j}$ decreases with j for each m.

Condition (1) implies that $|Ta|\leq T\,|a|$ and hence the non-negative sequences are sufficient to determine norm of T.

Proposition 1.1. ([5], Lemma 2.1). Let $p\geq 1$ and $T=(t_{i,j})$ be an operator with conditions (1), (2) and (3). Then

$$\|Ta\|_{d(w,p)}\leq\|Ta^*\|_{d(w,p)},$$

for all non-negative elements a in $d(w,p)$. Hence decreasing, non-negative elements are sufficient to determine norm of matrix operator T.

In the following, we state some lemmas which are needed for main result. We set $\xi^+=\max(\xi,0)$ and $\xi^-=\min(\xi,0)$ and also $p^*=p/_{p-1}$.

Lemma 1.1. ([2], Lemma 2.1). Assume that a,t are non-negative sequences. Then for all n

$$\sum_{k=1}^{n}t_k a_k\leq\left\{\max_{1\leq k\leq n}\frac{1}{n-k+1}\sum_{j=k}^{n}a_j\right\}\sum_{k=1}^{n}(n-k+1)(t_k-t_{k-1})^+.$$

Lemma 1.2. ([2], Lemma 2.2). Let $N\geq 1$ and a,t be non-negative sequences with $a_N\geq a_{N+1}\geq\cdots\geq 0$ and $a_n=0$ for $n<N$. Then for all n,

$$\sum_{k=1}^{n}t_k a_k\geq\left(\frac{1}{n}\sum_{j=1}^{n}a_j\right)\left\{nt_N+\frac{1}{n-N+1}\sum_{k=N+1}^{n}(n-k+1)(t_k-t_{k-1})^-\right\}.$$

Lemma 1.3. Suppose that u_n, v_n are non-negative numbers such that $\sum_{n=1}^{\infty} u_n$ is divergent and $\lim_{n\to\infty} v_n = 0$. Then

$$\frac{\sum_{n=1}^{m} u_n v_n}{\sum_{n=1}^{m} u_n} \to 0 \quad as \quad m \to \infty.$$

Proof. If we take $\varepsilon > 0$, since $\lim_{n\to\infty} v_n = 0$, then there exists an integer $N > 0$ such that for all $m > N$

$$\sum_{n=1}^{m} u_n v_n \le \sum_{n=1}^{N} u_n v_n + \varepsilon \sum_{n=N+1}^{m} u_n \le \sum_{n=1}^{N} u_n v_n + \varepsilon \sum_{n=1}^{m} u_n.$$

Since $\sum_{n=1}^{\infty} u_n$ is divergent, there exists an integer $N_1 > N$ such that for all $m > N_1$ we have

$$\sum_{n=1}^{N} u_n v_n \le \varepsilon \sum_{n=1}^{m} u_n.$$

Therefore

$$\sum_{n=1}^{m} u_n v_n \le 2\varepsilon \sum_{n=1}^{m} u_n.$$

If $\varepsilon \to 0$, we have the statement.

Proposition 1.2. ([5], Proposition 5.1). Let $p > 1$ and (w_n) be a decreasing sequence with non-negative entries and let the matrix $T = (t_{n,k})$ be with the following entries:

$$t_{n,k} = \begin{cases} \frac{1}{n} & for \quad n \ge k \\ 0 & for \quad n < k. \end{cases}$$

Then $\|T\|_{p,w} \le p^*$.

Lemma 1.4. Let $p > 1$ and (w_n) be a decreasing sequence with non-negative entries and also $\sum_{n=1}^{\infty} \frac{w_n}{n}$ be divergent. Let $N \ge 1$ and the matrix $C_N = (c_{n,k}^N)$ have the following entries:

$$c_{n,k}^N = \begin{cases} \dfrac{1}{n+N-1} & \text{for} \quad n \ge k \\ 0 & \text{for} \quad n < k \end{cases}.$$

Then $\|C_N\|_{p,w} = p^*$.

Proof. C_1 is the Cesaro matrix and $0 \le c_{n,k}^N \le c_{n,k}^1$ for all $n, k \ge 1$.

Since (w_n) is a decreasing sequence, applying Proposition 1.2, we deduce that

$$\|C_N\|_{p,w} \le \|C_1\|_{p,w} \le p^*.$$

Fix m such that $m \ge N$, and let

$$a_n = \begin{cases} (n+m-1)^{-1/p} & for \quad 1 \le n \le m \\ 0 & for \quad n > m, \end{cases}$$

then $\sum_{n=1}^{\infty} w_n a_n^p = \sum_{n=1}^{m} \frac{w_n}{n+m-1}$.

Also, for $n \le m$

$$A_n \ge \int_1^n (s+m-1)^{-1/p} ds = p^* \left((n+m-1)^{1/p^*} - m^{1/p^*} \right),$$

where $A_n = a_1 + \cdots + a_n$.

So that

$$b_n = \frac{A_n}{n+N-1} \ge \frac{p^*}{(n+m-1)^{1/p}} \left(1 - \left(\frac{m}{n+m-1} \right)^{1/p^*} \right).$$

Since $(1-s)^p \ge 1 - ps$ for $0 < s < 1$, we have

$$b_n^p \ge \frac{(p^*)^p}{n+m-1} \left(1 - p \left(\frac{m}{n+m-1} \right)^{1/p^*} \right),$$

and hence

$$\sum_{n=1}^{m} w_n b_n^p \ge (p^*)^p \sum_{n=1}^{m} \frac{w_n}{n+m-1}$$
$$- p (p^*)^p m^{\frac{1}{p^*}} \sum_{n=1}^{m} \frac{w_n}{(n+m-1)^{1+\frac{1}{p^*}}}.$$

Since (w_n) is a decreasing sequence, $w_n \ge w_{n+m-1}$ and so

$$\sum_{n=1}^{\infty} \frac{w_n}{n+m-1} \ge \sum_{n=1}^{\infty} \frac{w_{n+m-1}}{n+m-1} = \sum_{n=m}^{\infty} \frac{w_n}{n} = \infty.$$

Therefore $\sum_{n=1}^{\infty} \frac{w_n}{n+m-1}$ is divergent, setting $x_n = \frac{w_n}{n+m-1}$, $y_n = \frac{1}{(n+m-1)^{\frac{1}{p^*}}}$ and apply Lemma 1.3, we have the statement.

In the following, we recall Theorem 8 of [3] which is needed for main result.

Theorem 1.1. ([3], Theorem 8). If $p > 1$ and x is a non-negative sequence, then

$$\sum_{j=1}^{\infty}\max_{1\le i\le j}\left(\frac{1}{j-i+1}\sum_{k=i}^{j}x_k\right)^p \le \left(p^*\right)^p \sum_{1}^{\infty}x_k^p$$

Lemma 1.5. If $p>1$ and x, w are non-negative sequences and also w is decreasing, then

$$\sum_{j=1}^{\infty}w_j\max_{1\le i\le j}\left(\frac{1}{j-i+1}\sum_{k=i}^{j}x_k\right)^p \le \left(p^*\right)^p \sum_{k=1}^{\infty}w_k x_k^p$$

Proof. Applying Theorem 1.1, we have

$$\sum_{j=1}^{\infty}w_j\max_{1\le i\le j}\left(\frac{1}{j-i+1}\sum_{k=i}^{j}x_k\right)^p$$

$$\le \sum_{j=1}^{\infty}w_j\max_{1\le i\le j}\left(\frac{1}{j-i+1}\sum_{k=i}^{j}w_k^{1/p}x_k\right)^p$$

$$\le \left(p^*\right)^p \sum_{k=1}^{\infty}w_k x_k^p.$$

We set $t_{n,0}=0$ for $n\ge 1$ and

$$M_T=\sup_{n\ge 1}\left\{\sum_{k=1}^{n}(n-k+1)\left(t_{n,k}-t_{n,k-1}\right)^+\right\},$$

$$m_T=$$

$$\sup_{N\ge 1}\inf_{n\ge N}\left\{nt_{n,N}+\frac{n}{n-N+1}\sum_{k=N+1}^{n}(n-k+1)(t_{n,k}-t_{n,k-1})^-\right\}.$$

We say that $T=(t_{n,k})$ is a lower triangular, if $t_{n,k}=0$ for $n<k$. We now introduce the first main result.

Theorem 1.2. Suppose $p>1$ and (w_n) is a decreasing sequence with non-negative entries. Let $T=(t_{n,k})$ be a lower triangular matrix with non-negative entries.

(i) $\|T\|_{p,w}\le p^*M_T$. Moreover, if $M_T<\infty$, then T is bounded on $l_p(w)$.

(ii) If $\sum_{n=1}^{\infty}\frac{w_n}{n}$ is divergent and $(\frac{w_n}{w_{n+1}})$ is decreasing, then $\|T\|_{p,w}\ge p^*m_T$.

Therefore if (w_n) is a decreasing sequence with non-negative entries and $(\frac{w_n}{w_{n+1}})$ is decreasing and also $\sum_{n=1}^{\infty}\frac{w_n}{n}=\infty$, then

$$p^*m_T\le\|T\|_{p,w}\le p^*M_T.$$

Proof. (i) Let (a_n) be any sequence. By Lemma 1.1, we deduce that

$$\sum_{k=1}^{\infty}t_{n,k}a_k$$

$$\le\left\{\max_{1\le k\le n}\frac{1}{n-k+1}\sum_{j=k}^{n}a_j\right\}\sum_{k=1}^{n}(n-k+1)\left(t_{n,k}-t_{n,k-1}\right)^+$$

$$\le M_T\max_{1\le k\le n}\left\{\frac{1}{n-k+1}\sum_{j=k}^{n}a_j\right\}.$$

Applying Lemma 1.5 and the maximal theorem of Hardy and Littlewood, we have

$$\sum_{n=1}^{\infty}w_n\left(\sum_{k=1}^{\infty}t_{n,k}a_k\right)^p\le M_T^p\sum_{n=1}^{\infty}w_n\max_{1\le k\le n}\left(\frac{1}{n-k+1}\sum_{j=k}^{n}a_j\right)^p$$

$$\le\left(p^*M_T\right)^p\sum_{k=1}^{\infty}w_k a_k^p.$$

This implies that

$$\|T\|_{p,w}\le p^*M_T$$

(ii) We have $m_T=\sup_{N\ge1}\beta_N$, where

$$\beta_N$$

$$=\inf_{n\ge N}\left\{nt_{n,N}+\frac{n}{n-N+1}\sum_{k=N+1}^{n}(n-k+1)(t_{n,k}-t_{n,k-1})^-\right\}.$$

Let $N\ge1$, so that $\beta_N\ge0$. Let (b_n) be a decreasing sequence with non-negative entries and $\|b\|_{p,w}=1$. We set $a_1=\cdots=a_{N-1}=0$ and

$$a_{n+N-1}=\left(\frac{w_n}{w_{n+N-1}}\right)^{1/p}b_n,$$

for all $n\ge1$. We have $\|a\|_{p,w}=\|b\|_{p,w}=1$, and Lemma 1.2 follows that

$$\|T\|_{p,w}^p\ge\sum_{n=1}^{\infty}w_n\left(k=1\sum t_{n,k}a_k\right)^p$$

$$\ge\beta_N^p\sum_{n=1}^{\infty}w_n\left(\frac{1}{n}\sum_{j=1}^{n}a_j\right)^p$$

$$=\beta_N^p\sum_{n=1}^{\infty}w_{n+N-1}\left(\frac{1}{n+N-1}\sum_{j=1}^{n}a_{j+N-1}\right)^p$$

$$= \beta_N^p \sum_{n=1}^{\infty} w_{n+N-1} \left(\frac{1}{n+N-1} \sum_{j=1}^{n} \left(\frac{w_j}{w_j + N - 1} \right)^{1/p} b_j \right)^p$$

$$\geq \beta_N^p \|C_N b\|_{p,w}^p$$

Applying Proposition 1.1, we conclude that $\|T\|_{p,w} \geq p^* \beta_N$, and so

$$\|T\|_{p,w} \geq p^* m_T$$

This establishes the proof of the theorem.

In the following, we give some corollaries of Theorem 1.2. We assume (w_n) is a decreasing sequence with non-negative entries and $(\frac{w_n}{w_{n+1}})$ is decreasing and also $\sum_{n=1}^{\infty} \frac{w_n}{n} = \infty$.

Corollary 1.1. Suppose $p>1$ and $T=(t_{n,k})$ is a lower triangular matrix with $0 \leq t_{n,k-1} \leq t_{n,k}$ for $1<k \leq n$. Then

$$p^* \left(\sup_{N \geq 1} \inf_{n \geq N} nt_{n,N} \right) \leq \|T\|_{p,w} \leq p^* \left(\sup_{n \geq 1} \left\{ \sum_{k=1}^{n} t_{n,k} \right\} \right).$$

Moreover, if the right hand side of the above inequality is finite, then T is bounded on $l_p(w)$.

Proof. We have $M_T = \sup_{n\geq 1} \sum_{k=1}^{n} t_{n,k}$ and $m_T = \sup_{N\geq 1} \inf_{n \geq N} nt_{n,N}$. This completes the proof of the statement.

Corollary 1.2. Assume that $p>1$ and $T=(t_{n,k})$ is a lower triangular matrix with $0 \leq t_{n,k-1} \leq t_{n,k}$ for $1<k \leq n$ and also $(nt_{n,k})$ is an increasing sequence for each k. Then

$$\|T\|_{p,w} = p^* \left\{ \sup_{n \geq 1} nt_{n,n} \right\}.$$

In particular, $\|C_N\|_{p,w} = p^*$, where C_N is the generalized Cesaro matrix defined in Lemma 1.4.

We apply the above corollary to the following two special cases.

Let (t_n) be a non-negative sequence with $t_1>0$, and $T_n = t_1 + \cdots + t_n$. The Norlund matrix $N_t = (t_{n,k})$ is defined as follows:

$$t_{n,k} = \begin{cases} \dfrac{t_{n-k+1}}{T_n} & \text{for} \quad 1 \leq k \leq n \\ 0 & \text{for} \quad k > n \end{cases}.$$

Corollary 1.3. Suppose $p>1$ and $N_t=(t_{n,k})$ is the Norlund matrix and (t_n) is a sequence decreasing with $t_n \to \alpha$ and $\alpha > 0$. Then

$$\|N_t\|_{p,w} = p^*.$$

Let (t_n) be a non-negative sequence with $t_1>0$. The Weighted mean matrix $M_t = (t_{n,k})$ is defined as follows:

$$t_{n,k} = \begin{cases} \dfrac{t_k}{T_n} & \text{for} \quad 1 \leq k \leq n \\ 0 & \text{for} \quad k > n \end{cases}.$$

Corollary 1.4. Assume that $p>1$ and $M_t=(t_{n,k})$ is the Weighted mean matrix and also (t_n) is an increasing sequence with $t_n \to \alpha$ and $\alpha < \infty$. Then

$$\|M_t\|_{p,w} = p^*.$$

Corollary 1.5. Suppose $p>1$ and $T=(t_{n,k})$ is a lower triangular matrix with $t_{n,k-1} \geq t_{n,k} \geq 0$ for $1<k \leq n$. Then

$$p^* \left(\inf_{n \geq 1} \sum_{k=1}^{n} t_{n,k} \right) \leq \|T\|_{p,w} \leq p^* \left(\sup_{n \geq 1} \{ nt_{n,1} \} \right).$$

Moreover, if the right hand side of the above inequality is finite, then T is bounded on $l_p(w)$.

Proof. We have $M_T = \sup_{n\geq 1} nt_{n,1}$ and $m_T \geq \inf_{n\geq 1} \sum_{k=1}^{n} t_{n,k}$. This establishes the proof.

We apply the above corollary to the following two special cases.

Corollary 1.6. Assume that $p>1$ and $N_t=(t_{n,k})$ is the Norlund matrix and (t_n) is an increasing sequence. Then

$$p^* \leq \|N_t\|_{p,w} \leq p^* \left(\sup_{n \geq 1} \left\{ \frac{nt_n}{T_n} \right\} \right).$$

Corollary 1.7. Suppose $p>1$ and $M_t=(t_{n,k})$ is the Weighted mean matrix and also (t_n) is a decreasing sequence with $t_n\to\alpha$ and $\alpha>0$. Then

$$p^*\le\|M_t\|_{p,w}\le p^*\left(\frac{t_1}{\alpha}\right).$$

Example 1.1. Let $w_n=\frac{1}{(\log(n+1))^\gamma}$ where $0<\gamma\le1$, w_n and $(\frac{w_n}{w_{n+1}})$ be decreasing and also $\sum_{n=1}^{\infty}\frac{w_n}{n}=\infty$. Therefore, if (t_n) is a decreasing sequence with $t_n\to\alpha$ and $\alpha>0$, then

$$\|N_t\|_{p,w}=p^*.$$

Also, if (t_n) is an increasing sequence with $t_n\to\alpha$ and $\alpha<\infty$, then

$$\|M_t\|_{p,w}=p^*.$$

2. Matrix Operator on d(w, 1) and e(w,∞)

In this part of study, we consider the problem of finding the norm of matrix operator C_N and C_N^t on $d(w,1)$ and $e(w,\infty)$, where $d(w,1)$ and $e(w,\infty)$ are defined as before.

If $a\in d(w,1)$, we denote norm of a with $\|a\|_{1,w}$ and if $a\in e(w,\infty)$, we denote norm of a with $\|a\|_{w,\infty}$. We write $\|T\|_{1,w}$ for the norm of T as an operator from $d(w,1)$ into itself, and $\|T\|_{w,\infty}$ for the norm of T as an operator from $e(w,\infty)$ into itself.

Suppose T is a bounded matrix operator on $e(w,\infty)$. Then T^t, the transpose matrix of T, is a bounded matrix operator on $d(w,1)$ and

$$\|T^t\|_{1,w}=\|T\|_{w,\infty}.$$

Let $N\ge1$ and C_N be defined as in Lemma 1.4, and also let C_N^t be the matrix transpose of C_N. The matrix $C_N^t=(a_{n,k})$ is defined as follows:

$$a_{n,k}=\begin{cases}\dfrac{1}{k+N-1} & \text{for} \quad n\le k\\ 0 & \text{for} \quad n>k\end{cases}.$$

If $N=1$, C_1 and C_1^t are Cesaro and Copson matrices, respectively. C_N and C_N^t are generalized Cesaro and Copson matrices.

The problem of finding the norm of matrix operators on $d(w,1)$ and $e(w,\infty)$ is considered in [8]. Also in the following, we consider such problems for some matrices on weighted sequence spaces $d(w,1)$ and $e(w,\infty)$.

Theorem 2.1. Suppose $T=(t_{n,k})$ is a matrix operator satisfying conditions (1), (2) and (3). If

$$\sup_n\frac{S_n}{W_n}<\infty,$$

where $S_n=s_1+\cdots+s_n$, $s_n=\sum_{k=1}^{\infty}w_k t_{k,n}$, and $W_n=w_1+\cdots+w_n$, then T is a bounded operator from $d(w,1)$ into itself, and also

$$\|T\|_{1,w}=\sup_n\frac{S_n}{W_n}.$$

Proof. Applying Proposition 1.1, it is sufficient to consider decreasing, non-negative sequences. Let a be in $d(w,1)$ such that $a_1\ge a_2\ge\cdots\ge0$ and $M=\sup_n\frac{S_n}{W_n}$. Then

$$\begin{aligned}\|Ta\|_{1,w}&=\sum_{n=1}^{\infty}w_n\left(\sum_{k=1}^{\infty}t_{n,k}a_n\right)\\&=\sum_{n=1}^{\infty}s_n a_n\\&=\sum_{n=1}^{\infty}S_n\left(a_n-a_{n+1}\right)\\&\le M\sum_{n=1}^{\infty}W_n\left(a_n-a_{n+1}\right).\end{aligned}$$

Also, we have

$$\|a\|_{1,w}=\sum_{n=1}^{\infty}W_n\left(a_n-a_{n+1}\right).$$

Therefore

$$\|Ta\|_{1,w}\le M\,\|a\|_{1,w},$$

and hence $\|T\|_{1,w}\le M$.

Further, we take $a_1=\cdots=a_n=1$ and $a_k=0$ for all $k\ge n+1$, then

$$\|a\|_{1,w}=W_n,\qquad \|Ta\|_{1,w}=S_n$$

Thus

$\|T\|_{1,w} = M$.

This completes the proof of the theorem.

In the following statements, we consider the norm of Cesaro and Copson matrices. It is enough to consider the sequence $(\frac{s_n}{w_n})$ instead of $(\frac{S_n}{W_n})$, because of the well-known facts listed in the following lemma.

Lemma 2.1. (i) If $m \le \frac{s_n}{w_n} \le M$ for all n, then $m \le \frac{S_n}{W_n} \le M$ for all n.

(ii) If $(\frac{s_n}{w_n})$ is increasing (or decreasing), then so is $(\frac{S_n}{W_n})$.

(iii) If $\frac{s_n}{w_n} \to M$ as $n \to \infty$, then $\frac{S_n}{W_n} \to M$ as $n \to \infty$.

Proof. It is elementary.

Lemma 2.2. Let $0 < \alpha < 1$.

(i) If $N \ge 1$ and $X_n = \sum_{k=1}^{n} \frac{1}{(k+N-1)^{\alpha}}$, then $\frac{X_n}{(n+N-1)^{1-\alpha}}$ is increasing and tends to $\frac{1}{1-\alpha}$.

(ii) If $X_{(n)} = \sum_{k=n}^{\infty} \frac{1}{k^{1+\alpha}}$, then $n^{\alpha} X_{(n)}$ is decreasing.

Proof. It is elementary.

Theorem 2.2. If $w_n = \frac{1}{(n+N-1)^{\alpha}}$, where $0 < \alpha < 1$, then C_N is a bounded operator on $d(w,1)$ and also C_N^t is a bounded operator on $e(w,\infty)$. Moreover,

$$\|C_N\|_{1,w} = \|C_N^t\|_{w,\infty} = N^{\alpha} \sum_{k=1}^{\infty} \frac{1}{(k+N-1)^{1+\alpha}}.$$

In particular, $\|C_1\|_{1,w} = \|C_1^t\|_{w,\infty} = \xi(1+\alpha)$, where ξ is Riemann's Zeta function.

Proof. Applying Theorem 2.1, we have

$$\|C_N\|_{1,w} = \sup_n \frac{S_n}{W_n}.$$

Since

$$\frac{s_n}{w_n} = (n+N-1)^{\alpha} \sum_{k=n}^{\infty} \frac{1}{(k+N-1)^{1+\alpha}}$$

$$= (n+N-1)^{\alpha} \sum_{k=n+N-1}^{\infty} \frac{1}{k^{1+\alpha}},$$

Lemma 2.2 (ii) shows that $\frac{s_n}{w_n}$ is decreasing. Therefore applying Lemma 2.1 (ii), we deduce that $\frac{S_n}{W_n}$ is decreasing and also

Proposition 2.1. If

$$r_N(w) = \sup_{n \ge 1} \frac{W_n}{(n+N-1)w_n} < \infty,$$

then C_N^t maps $d(w,1)$ into itself. Also, we have

$$\|C_N^t\|_{1,w} \le r_N(w).$$

Proof. Since for all n

$$s_n = \frac{W_n}{n+N-1} \le r_N(w) w_n,$$

Theorem 2.1 and Lemma 2.1 (i) follow that $\|C_N^t\|_{1,w} \le r_N(w)$, and this completes the proof.

Proposition 2.2. If

$$\sup_{n \ge 1} \frac{1}{W_n} \sum_{k=1}^{n} \frac{W_k}{k+N-1} < \infty,$$

then C_N is a bounded operator on $e(w,\infty)$ and

$$\|C_N\|_{w,\infty} = \sup_{n \ge 1} \frac{1}{W_n} \sum_{k=1}^{n} \frac{W_k}{k+N-1}.$$

Proof. Applying Theorem 2.1, we have

$$\|C_N^t\|_{1,w} = \sup_n \frac{S_n}{W_n}.$$

Since $s_n = \frac{W_n}{n+N-1}$, and $\|C_N^t\|_{1,w} = \|C_N\|_{w,\infty}$, we have the statement.

Theorem 2.3. Suppose that $w_n = \frac{1}{(n+N-1)^{\alpha}}$, where $0 < \alpha < 1$. Then C_N^t maps $d(w,1)$ into itself and also we have

$$\|C_N\|_{w,\infty} = \|C_N^t\|_{1,w} = \frac{1}{1-\alpha}.$$

In particular, $\|C_1\|_{w,\infty} = \|C_1^t\|_{1,w} = \frac{1}{1-\alpha}$.

Proof. We have

$$\frac{S_n}{w_n} = \frac{W_n}{(n+N-1)w_n} = \frac{W_n}{(n+N-1)^{1-\alpha}}.$$

Our W_n is the X_n of Lemma 2.2 (i), which tells us that $\frac{W_n}{(n+N-1)^{1-\alpha}}$ is increasing and tends to $\frac{1}{1-\alpha}$. Lemma 2.1 (ii) and (iii) follow the statement (Of course, this also shows that $r_N(w) = \frac{1}{1-\alpha}$.).

References

1. Bennett G. Inequalities complimentary to Hardy, Quart. J. Math. Oxford (2), **49:** 395-432 (1998).
2. Chang-Pao Chen, Dah-Chin Luor, and Zong-Yin Ou, Extension of Hardy inequality, J. Math. Appl., **237:** 160-171 (2002).
3. Hardy G.H., and Littlewood J.E. A maximal theorem with function-theoretic, Acta Math. **54:** 81-116 (1930).
4. Hardy G.H., Littlewood J.E., and Polya G. Inequalities, 2nd edition, Cambridge University press, Cambridge (2001).
5. Jameson G.J.O., and Lashkaripour R. Norms of certain operators on weighted lp spaces and Lorentz sequence spaces, J. Inequalities in Pure and Applied Mathematics. Volume **3**, Issue 1, Article 6 (2002).
6. Lashkaripour R. Weighted Mean Matrix on Weighted Sequence Spaces, WSEAS Transaction on Mathematics. Issue 4, Volume **3:** 789-794 (2004).
7. Lashkaripour R. Transpose of the Weighted Mean Operators on Weighted Sequence Spaces, WSEAS Transaction on Mathematics. Issue 4, Volume **4:** 380-385 (2005).
8. Lashkaripour R., and Foroutannia D. Inequalities involving upper bounds for certain matrix operators, Proc. Indian Acad. Sci.(Math. Sci.) Volume **116:** 325-336 Agust (2006).
9. Lashkaripour R., and Foroutannia D. Lower bounds for matrices on weighted sequence spaces, Journal of Sciences, Islamic Republic of Iran **18**(1): 49-56 (2007).
10. Lashkaripour R., and Foroutannia D. Some inequalities involving upper bounds for some matrix operators I, Czech. Math. J. **57**(132): 553-572 (2007).