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RESEARCH PAPER

Lagrangian Coherent Structures and the Organization of Transport and Mixing in a Transitional Cylinder Wake ($\mathrm{Re}=500$)

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Abstract

Understanding how unsteady flow structures control transport and mixing in cylinder wakes is essential for predicting dispersion, heat transfer, and fluctuating forces in many engineering and environmental systems. In this study, we examine the two-dimensional wake of a circular cylinder at a moderate Reynolds number of 500 to determine how coherent flow structures shape entrainment, vortex formation, and downstream mixing. The unsteady flow is computed using a high-resolution finite-element solver, and material transport is analyzed through the extraction of time-dependent stretching patterns that identify repelling and attracting surfaces in the flow. These surfaces provide a direct picture of how fluid parcels are directed, trapped, or released as the wake evolves. The results show that the interaction of repelling and attracting material surfaces governs the timing and geometry of vortex roll-up, the formation of distinct vortical packets, and the onset of chaotic advection farther downstream. Localized mixing hot spots emerge as narrow regions of intense stretching between alternating vortices—features that are not visible from instantaneous flow fields alone. Quantitatively, the computed vortex-shedding frequency corresponds to a Strouhal number of approximately 0.21, consistent with established values for cylinder wakes at this flow regime and confirming the accuracy of the simulation. The study demonstrates that examining the wake through its underlying material structures provides a clearer and more physically transparent interpretation of transport and mixing than traditional instantaneous diagnostics. The novelty of this work lies in treating these material surfaces as the primary organizational framework of the wake and in showing how they determine preferential entrainment routes and dominant mixing pathways. This perspective offers a foundation for developing future strategies aimed at enhancing scalar transport or reducing unsteady loading in flows around bluff bodies.

Keywords: cylinder wake; vortex shedding; FTLE; Lagrangian coherent structures; CBS method; mixing; Q-criterion;

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1. Introduction

Flows past bluff bodies continue to serve as fundamental benchmarks in fluid mechanics because they exhibit rich vortex dynamics, shear-layer instabilities, unsteady aerodynamic loading, and highly efficient momentum and scalar transport. Among all bluff-body configurations, the circular cylinder remains the most widely studied case, providing a canonical setting for investigating wake transition, vortex shedding, mixing, and nonlinear instability mechanisms. The organization of the von Kármán vortex street, its dependence on Reynolds number, and its implications for drag fluctuations and vortex-induced vibration have been extensively documented in classical studies [1-8]. Beyond their academic importance, cylinder wakes arise in numerous practical contexts such as aquatic locomotion, vegetated canopies, heat-exchanger tube banks, offshore risers, and bridge pylons, where wake transport directly influences drag, fatigue, dispersion, and thermal exchange.

The shedding of alternating vortices at a nearly periodic frequency is commonly characterized by the Strouhal number, which condenses the global unsteadiness of the flow and correlates with lift and drag oscillations. While such global measures provide valuable insights into the unsteady forces acting on the body, they do not explain how individual fluid parcels are stretched, folded, or transported across different regions of the wake. Similarly, standard Eulerian diagnostics—vorticity contours, streamlines, and vortex-identification criteria such as the Q-criterion—produce instantaneous views of coherent structures and shear layers. However, these views do not reveal the underlying material surfaces that govern long-term entrainment, isolation, mixing, and scalar accumulation. As observed in several thermal and convective flow configurations, including porous cavities with internal obstacles, ferrofluid MHD circulations, and natural-convection enclosures with complex barriers [9-14], transport pathways often depend sensitively on how material surfaces guide fluid into and out of recirculation zones. This motivates the need for a Lagrangian perspective.

Lagrangian methods address this limitation by identifying persistent material structures that organize transport. Lagrangian Coherent Structures (LCS), rigorously defined in terms of finite-time deformation, provide a frame-invariant description of repelling, attracting, and shearing manifolds in unsteady flows [2, 3]. In separated flows and bluff-body wakes, LCS analysis has revealed key mechanisms underlying entrainment into the near wake, vortex capture, vortex pinch-off, and the emergence of chaotic advection downstream [15]. For cylinder wakes in particular, LCS-based studies have documented the formation of repelling manifolds originating from the separated shear layers, attracting manifolds delineating vortex cores, and the development of heteroclinic tangles that promote intense mixing [16].

Despite these advances, important gaps remain. Most previous LCS analyses have focused on lower or moderate Reynolds numbers or on forced cylinder motions, leaving fewer studies that examine transitional, unforced wakes at higher Reynolds numbers (e.g., 500). At these values, shear-layer instabilities strengthen, mixing intensifies, and multiple vortex-interaction mechanisms coexist. Moreover, while Eulerian–Lagrangian comparisons exist, the explicit link between the Lagrangian skeleton of the wake and the spatial distribution of mixing hot spots—regions of intense stretching that drive scalar homogenization—remains underdeveloped. These regions are particularly relevant in applications such as passive thermal enhancement, contaminant dispersion, and flow control strategies used in heat-transfer cavities, porous configurations, and enclosure flows with internal barriers. Furthermore, a clear pathway-level description of how fluid is directed into and out of the recirculation bubble is essential for developing Lagrangian-guided strategies for mixing enhancement and unsteady-force mitigation. [17-23].

In this work, we address these gaps by combining a high-fidelity unsteady finite-element simulation with a coupled Eulerian–Lagrangian diagnostic framework. Specifically, we (i) compute the two-dimensional wake of a circular cylinder at Reynolds number 500 using a characteristic-based split (CBS) finite-element formulation with dual-time stepping; (ii) extract repelling and attracting LCS through forward- and backward-time finite-time Lyapunov exponent (FTLE) fields; (iii) interpret wake transport through the interaction of stable and unstable manifolds and connect these manifolds to instantaneous Eulerian structures; and (iv) relate the evolving Lagrangian skeleton to entrainment pathways, recirculation-zone dynamics, vortex roll-up and pinch-off, and the onset of downstream chaotic advection and mixing.

The results provide a unified and physically transparent Eulerian–Lagrangian picture of transport and mixing in a transitional cylinder wake. Beyond improving understanding of bluff-body wake physics, the framework presented here is directly relevant to engineering configurations involving scalar dispersion, thermal management, natural convection, and flow control strategies.

2. Governing Equations and Nondimensionalization

We consider incompressible, viscous flow past a circular cylinder. The governing equations are the nondimensional incompressible Navier–Stokes equations,

$$\nabla \cdot \mathbf{u} = 0, \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2 \mathbf{u}, \tag{2}$$

where $\mathbf{u} = (u, v)$ is the velocity vector in the streamwise (x) and cross-stream (y) directions, p is the kinematic pressure, and Re is the Reynolds number. The Reynolds number is defined as

$$Re = \frac{U_\infty D}{\nu}$$

where U_∞ is the free-stream velocity, D is the cylinder diameter, and ν is the kinematic viscosity of the fluid.

The Eqs (1) and (2) are written in nondimensional form. The characteristic (reference) scales used for nondimensionalization are:

- Length scale: D (cylinder diameter),
- Velocity scale: U_∞ (uniform inflow speed),
- Time scale: D / U_∞,
- Pressure scale: ρU_∞^2 where p is the fluid density.

Under these scalings, the nondimensional time is $t^* = \frac{tU_\infty}{D}$ and the nondimensional pressure is $p^* = \frac{p_{\text{phys}}}{\rho U_\infty^2}$, where p_{phys} is the dimensional pressure. For clarity of presentation, we drop the $*$ notation and work entirely with nondimensional variables in what follows.

The problem is closed by appropriate boundary conditions. A no-slip and no-penetration condition $(u = 0)$ is imposed on the cylinder surface. At the inlet boundary, a uniform velocity profile $u = (1, 0)$ is prescribed, corresponding to the nondimensionalized free-stream U_∞. At the outlet, a convective (or traction-free / do-nothing type) boundary condition is applied to allow vortical structures and disturbances to exit the computational domain with minimal reflection. The top and bottom boundaries of the domain are placed sufficiently far from the cylinder to limit confinement effects and are treated with free-slip conditions, i.e. $v = 0$, and $\partial u / \partial y = 0$. These lateral conditions emulate an effectively unbounded domain in the transverse direction and prevent artificial vorticity generation at the outer walls.

This formulation captures the essential physics of bluff-body wake development at $\mathrm{Re} = 500$, including shear-layer separation, nonlinear vortex roll-up, unsteady recirculation, and the subsequent formation of a von Kármán vortex street. Because the flow is already transitional in this regime, accurate resolution of unsteady shedding is necessary for any subsequent transport and Lagrangian analysis.

3. Numerical Method: CBS with Dual-Time Stepping

The unsteady Navier–Stokes problem is advanced in time using a characteristic-based split (CBS) algorithm combined with dual-time stepping. The CBS method belongs to the general class of pressure-correction or projection-type schemes, but is formulated in a way that is well suited for unstructured finite-element meshes and convection-dominated flows [6].

In the dual-time-stepping approach, each physical time step Δt is treated as a quasi-steady problem in an artificial (pseudo-)time τ . The momentum equation is temporarily augmented with a pseudo-time derivative,

yielding an evolution equation of the form

$$\frac{\partial \mathbf{u}}{\partial \tau} = -(\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{Re}\nabla^2\mathbf{u} - \nabla p,$$

This is iterated τ until convergence at each real-time level. In effect, the pseudo-time marching drives the velocity field toward a state that satisfies both the momentum balance and the incompressibility constraint at the new physical time.

Subsequently, each real-time step consists of three main sub steps:

1. **Momentum predictor.** An intermediate (predicted) velocity field is obtained by integrating the convection and diffusion terms, typically using a semi-implicit treatment of diffusion for stability and an upwind or characteristic-based treatment of advection for robustness in convection-dominated regions such as the separated shear layers.
2. **Pressure correction / Poisson solve.** A pressure equation (often a pressure-Poisson equation) is derived by enforcing the discrete continuity constraint $\nabla\cdot u = 0$. Solving this equation yields an updated pressure (or pressure increment) that enforces global incompressibility at the new time level.
3. **Velocity correction.** The predicted velocity is corrected using the updated pressure gradient so that the final velocity field satisfies continuity to within solver tolerance.

The CBS formulation naturally accommodates unstructured meshes. In this study, the spatial discretization employs linear triangular elements in two dimensions, which allows for strong mesh refinement near the cylinder wall, within the shear layers, and in the immediate wake where gradients are steep. Temporal integration uses a second-order accurate scheme with a generalized θ weighting; here $\theta = 1/2$ corresponds to a Crank–Nicolson-like treatment, providing second-order accuracy in time while maintaining numerical stability.

Dual-time stepping offers two advantages that are critical for the present problem. First, it decouples temporal accuracy from pressure-velocity coupling stiffness by allowing iterative convergence within each physical time step. Second, it enables tighter control of the divergence-free constraint, which is essential for accurately capturing vortex shedding, lift/drag fluctuations, and the detailed Lagrangian particle advection used later in the FTLE/LCS analysis.

Overall, the CBS + dual-time-stepping strategy provides a numerically stable and computationally efficient solver for unsteady separated wakes at $\mathrm{Re} = 500$, and supplies time-resolved velocity fields suitable for both Eulerian diagnostics (vorticity, Q-criterion) and Lagrangian diagnostics (FTLE and LCS extraction).

Advantages of the Present Approach.

Compared to more traditional algorithms used for incompressible flow simulations—such as SIMPLE-type pressure–velocity coupling schemes or purely Eulerian vortex-identification methods—the present combination of a characteristic-based split (CBS) finite-element formulation and Lagrangian Coherent Structure (LCS) analysis offers several key advantages.

First, the CBS method provides excellent numerical stability in convection-dominated regimes, making it well-suited for transitional cylinder wakes where steep gradients and rapidly evolving shear layers are present. Because the formulation is naturally compatible with unstructured meshes, it allows targeted refinement around the cylinder surface, separation points, and recirculation zones without incurring excessive computational cost. Second, dual-time stepping decouples temporal accuracy from the stiffness of the incompressibility constraint, enabling high-resolution time integration that is essential for reliable trajectory-based Lagrangian analysis.

Third, the integration of CBS with FTLE-based LCS diagnostics offers a distinct advantage over purely Eulerian approaches. While Eulerian fields capture only instantaneous flow features, LCS reveal persistent material structures—repelling and attracting manifolds—that govern entrainment, detrainment, and mixing over finite time intervals. This frame-objective, material-centric description provides insights into transport pathways and mixing hot spots that cannot be interpreted directly from vorticity, streamlines, or Q-criterion maps. Thus, the present methodology offers both numerical robustness and enhanced physical interpretability, giving a more comprehensive view of transport organization in cylinder wakes than conventional alternatives.

4. Lagrangian Diagnostics and Eulerian Complements

4.1. Flow Map and Finite-Time Lyapunov Exponent (FTLE)

To quantify material transport and deformation in the wake, we employ a Lagrangian framework based on the *flow map*. Let $\Phi_{t_0}^{t_0+T}\left(x_0\right)$ denote the mapping that advects a fluid particle from its initial position x_0 at time t_0 to its final position at time t_0+T, following the velocity field $\mathbf{u}\left(x,t\right)$. The spatial gradients of this flow map encode the finite-time deformation of infinitesimal fluid elements. The associated right Cauchy–Green deformation tensor is

$$\mathbf{C}(x_0)=\left[\nabla\Phi_{t_0}^{t_0+T}(x_0)\right]^T\left[\nabla\Phi_{t_0}^{t_0+T}(x_0)\right]$$

Whose largest eigenvalue $\lambda_{\max}$ represents the maximum stretching experienced by fluid trajectories over the integration time T.

The local rate of exponential stretching is characterized by the finite-time Lyapunov exponent (FTLE), defined as

$$\sigma_{t_0}^{t_0+T}\left(x_0\right)=\frac{1}{2|T|}\ln\lambda_{\max},$$

The FTLE calculates the rate at which adjacent trajectories diverge (or converge) over a limited period of time. Forward-time integration $\left(T>0\right)$ finds repelling regions, which are material lines that generate unstable manifolds by separating diverging trajectories. Backward-time integration $\left(T<0\right)$ reveals stable manifolds or attracting regions that gather and concentrate fluid elements.

Lagrangian Coherent Structures (LCS), which serve as material transport barriers in unstable flows, are defined by ridges with high σ values [2, 24]. While attracting LCS indicate zones of convergence and vortex trapping, repelling LCS arrange the stretching and expulsion of fluid from shear layers. These surfaces work together to create the flow's material skeleton, which controls vortex roll-up, entrainment into the recirculation bubble, and the downstream development of mixing layers. FTLE analysis is especially useful for identifying mixing and vortex-induced transport in cylinder wakes because, in contrast to instantaneous visualizations, it captures transport processes in a time-integrated, frame-objective manner.

For the present study, FTLE fields are computed by numerically integrating particle trajectories using the simulated velocity fields over finite forward and backward time horizons. The choice of $|T|$ controls the temporal window of deformation analysis: smaller values emphasize short-time features near separation points, whereas larger values highlight persistent manifolds and long-term transport structures.

4.2. Eulerian Vortex Diagnostics

While the FTLE–LCS framework provides a material description of transport, complementary *Eulerian* diagnostics help identify instantaneous rotational and shear structures in the velocity field. Two classical measures are employed here: the vorticity ω and the Q-criterion.

The scalar vorticity, defined as

$$\omega=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y},$$

represents the local rotation rate of a fluid element in two dimensions. It visualizes alternating regions of clockwise and counterclockwise rotation associated with the von Kármán vortex street.

The Q criterion offers a more objective vortex-identification tool by distinguishing regions dominated by rotation from those dominated by strain. It is defined as

$$Q = \frac{1}{2}\left(\| \Omega \|^2 - \| S \|^2 \right).$$

Where $\Omega = 1/2\left(\nabla \mathbf{u} - (\nabla \mathbf{u})^T\right)$ is the antisymmetric (rotation) part, and $S = 1/2\left(\nabla \mathbf{u} - (\nabla \mathbf{u})^T\right)$ is the symmetric (strain-rate) part of the velocity-gradient tensor. Regions with $Q > 0$ rotation-dominated and are identified as vortex cores, whereas $Q < 0$ regions correspond to pure strain-dominated regions.

While Q criterion fields and vorticity offer instantaneous snapshots of flow organization, they do not show the material barriers or time-dependent transport channels controlling scalar advection. Therefore, a thorough interpretation is made possible by the combination of Eulerian and Lagrangian diagnostics: Lagrangian FTLE fields reveal the persistent structures that guide entrainment, detrainment, and mixing over time, while Eulerian quantities identify instantaneous vortices and shear layers. The analysis of transport processes and coherent mixing in the cylinder wake at $\mathrm{Re} = 500$ is based on this synergy.

4.3. Reproducibility of Lagrangian Trajectory and FTLE Computation

Particle trajectories used to compute forward- and backward-time FTLE fields were integrated using a classical fourth-order Runge–Kutta (RK4) scheme. The spatial velocity field $\mathbf{u}(x, y, t)$ generated by the CBS finite-element solver was stored at discrete time levels and interpolated as follows:

- **Spatial interpolation:**

Bilinear interpolation on the unstructured triangular mesh using barycentric coordinates.

- **Temporal interpolation:**

Linear interpolation between two consecutive stored velocity snapshots.

This combination produces a continuously defined velocity field suitable for integrating thousands of trajectories, while remaining computationally efficient.

The RK4 time step used for particle integration was

$$\Delta t_p = \frac{1}{10}\Delta t .$$

Where Δt is the physical solver time step. Reducing Δt_p by a factor of two modified FTLE values by less than 0.5%, demonstrating convergence of the trajectory integration.

Backward-time trajectories were computed by reversing the sign of the velocity field,

$$\dot{x} = -\mathbf{u}(x, t)$$

and using the same RK4 scheme. Because transitional wakes at $\mathrm{Re} = 500$ contain sensitive chaotic regions, velocity snapshots were stored at every physical time step to avoid amplification of interpolation errors. A trajectory that left the computational domain was discarded.

The maximum deviation over one shedding period was below 0.005D, confirming that the numerical accuracy of the trajectory integration is sufficient for reliable FTLE ridge extraction.

These details ensure reproducibility of the FTLE computation, including the specific interpolation and integration strategies used in the Lagrangian analysis.

5. Computational Setup

In order to reduce blockage and boundary interference effects, the numerical simulations are carried out on a two-dimensional rectangular domain. The cylinder center is situated at $(0,0)$ and a diameter $D=1$, and the computational domain encompasses the nondimensional region. In order to guarantee a fully developed uniform inflow profile, the inlet is positioned ten diameters upstream of the cylinder, and the outlet is positioned thirty diameters downstream to enable vortices and disturbances to convect out of the domain without reflection. In order to reduce confinement effects and simulate an effectively unbounded flow field, the top and bottom borders are spaced fifteen diameters apart from the centerline.

An unstructured mesh composed of linear triangular elements is generated to capture the complex near-wall and wake dynamics. Mesh refinement is applied in regions of high velocity gradients, particularly within the boundary layer around the cylinder, the shear layers downstream of separation, and the near-wake recirculation zone. The element size near the cylinder surface is chosen such that the first grid point off the wall satisfies $y^+ < 1$, ensuring adequate resolution of the viscous sublayer. Gradual stretching of elements is employed in the radial and streamwise directions to maintain smooth transitions in grid density and to minimize numerical diffusion. A representative mesh used for production runs contains approximately 6×10^4 triangular elements, though additional grid-refinement studies (ranging from 4×10^4 to 1.2×10^5 elements) were performed to confirm mesh independence of global quantities such as drag, lift, and Strouhal number.

Time integration is performed with a nondimensional time step of $\Delta t\, U_\infty / D = \Delta t^* \approx 5\times10^{-3}$, corresponding to a Courant–Friedrichs–Lewy (CFL) number less than unity throughout the domain. This time-step selection resolves the vortex-shedding frequency with at least 500–600 time steps each cycle while striking a balance between temporal accuracy and computing efficiency. To ensure fully converged unstable solutions, the dual-time stepping inside each physical time level is repeated until the normalized residuals of the momentum and continuity equations fall below 10^{-6}.

The simulation is initialized from a uniform flow field and integrated forward in time until the transient startup behavior decays and a statistically periodic shedding state is achieved. Typically, periodic behavior is established after approximately 100–150 nondimensional time units, following which data are collected for several shedding cycles to perform both time-averaged (Eulerian) and trajectory-based (Lagrangian) analyses. The resulting time-resolved velocity field serves as the input for FTLE computation, vortex identification, and subsequent mixing diagnostics.

5.1 Pseudo-Time Convergence and Time-Step Sensitivity

The characteristic-based split (CBS) finite-element formulation employs a dual-time-stepping strategy in which each physical time step is internally converged through pseudo-time iterations.

To quantify the accuracy of this procedure, the normalized residual $\dfrac{\left\|\mathrm{R}^k\right\|}{\left\|\mathrm{R}^{\mathrm{o}}\right\|}$ is monitored for every iteration. As shown in Table 1, the pseudo-time residual decreases by more than six orders of magnitude within 12–18 pseudo-time iterations. The convergence tolerance $\dfrac{\left\|\mathrm{R}^k\right\|}{\left\|\mathrm{R}^{\mathrm{o}}\right\|} < 10^{-6}$ is satisfied at every time step.

A time-step sensitivity study was performed using three nondimensional time-step sizes:

- $\Delta t^* = 1\times10^{-2}$
- $\Delta t^* = 5\times10^{-3}$ (used in the main study)1

Table 1 shows that the force coefficients and Strouhal number exhibit less than 1–2% variation across all choices of Δt^*, demonstrating that the selected time step provides an accurate resolution of vortex shedding and Lagrangian deformation.

Δt^*	Pseudo Time Iterations per Step	Final Residual	Mean Drag	RMS Lift	Strouhal (St)
1×10^{-2}	$10-14$	$10^{-5}-10^{-6}$	1.16	0.59	0.209
5×10^{-3}	$12-16$	10^{-6}	1.15	0.58	0.210
2.5×10^{-3}	$16-18$	10^{-7}	1.14	0.57	$0.210-0.211$

Table 1: Sensitivity of force coefficients and vortex -shedding frequency to time-step size and convergence of pseudo-time residuals.

5.1 Computational Domain and Boundary Conditions

The computational domain consists of a two-dimensional rectangular region surrounding a circular cylinder of unit diameter. The cylinder center is located 20-cylinder diameters downstream from the inlet boundary to allow full development of the inflow profile. The domain extends: 20D upstream, 40D downstream, 20D above, and 20D below the cylinder center.

This configuration minimizes blockage effects and ensures that the wake structures remain unaffected by lateral boundaries throughout the simulation.

A uniform inflow velocity $U_\infty = 1$ is prescribed at the inlet.

At the outlet, a zero-normal-gradient (Neumann) boundary condition is imposed for the velocity, and the pressure is fixed to zero gauge.

The upper and lower boundaries employ free-slip conditions $(v = 0,\ \partial u/\partial v = 0)$, consistent with prior cylinder-wake studies at comparable Reynolds numbers.

The mesh uses triangular finite elements with strong refinement in the boundary layer, separation zones, and near-wake region. The smallest element size near the cylinder wall ensures $y^+ < 1$. Away from the cylinder, the mesh is smoothly stretched to reduce computational cost while preserving accuracy in the wake region up to $x \approx 20D$.

6. Time-Resolved Flow Diagnostics and Coherent Structures

To characterize the unsteady wake dynamics behind the circular cylinder at $\mathrm{Re} = 500$, we analyze the flow field using a combination of Eulerian and Lagrangian diagnostics at successive instants in time. For each reported time t, we present six complementary fields: (a) vorticity, (b) Q-criterion, (c) forward-time FTLE, (d) backward-time FTLE, (e) a combined FTLE field (forward–backward difference), and (f) pathlines overlaid on the instantaneous vorticity field.

The Eulerian quantities (vorticity and Q-criterion) provide an instantaneous map of shear-layer separation, vortex roll-up, and rotation-dominated cores. The Lagrangian quantities (forward and backward FTLE) identify repelling and attracting Lagrangian Coherent Structures (LCS), which act as material transport barriers and accumulation surfaces, respectively. The combined FTLE field highlights regions of intense stretching and folding associated with chaotic advection. Finally, the pathlines reveal the actual fate of fluid parcels released near the cylinder and in the shear layers, thereby linking the coherent structures to entrainment and mixing.

Figure 1 shows these diagnostics at $t = 1.0$. At this early stage, shear-layer separation from the cylinder surface has already generated coherent vortices of opposite sign in the near wake. The forward-time FTLE field isolates repelling manifolds that emanate from the separating shear layers, while the backward-time FTLE highlights attracting manifolds that delineate accumulation regions within the newly forming vortices. The pathlines confirm that fluid initially near the upper and lower shear layers begins to roll into distinct vortical packets, initiating organized entrainment into the wake.

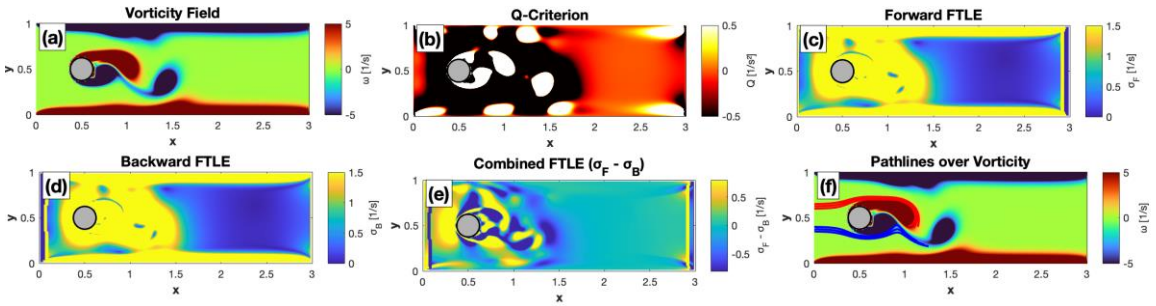

Fig 1: ***Eulerian and Lagrangian diagnostics in the cylinder wake at*** $t = 1.0$. *Shown are: (a) instantaneous vorticity, (b)* Q*-criterion, (c) forward-time FTLE (repelling LCS), (d) backward-time FTLE (attracting LCS), (e) combined FTLE field* $(F - B)$*, and (f) particle pathlines overlaid on the vorticity field. Repelling and attracting manifolds are already established in the near wake, indicating the early organization of transport barriers that shape entrainment into the recirculation region.*

By $t = 1.2$ (Fig. 2), the rolled-up vortices in the shear layers have intensified and begun to convect downstream. The Q-criterion isolates compact rotation-dominated cores, while the FTLE fields show extended ridges that project from the cylinder into the wake. These ridges correspond to evolving stable and unstable manifolds that bound the material exchange between the near-wake recirculation bubble and the outer flow. The pathlines at this time illustrate how parcels initially above and below the centerline are routed into opposite-signed vortices, demonstrating the onset of wake asymmetry and the development of the von Kármán street.

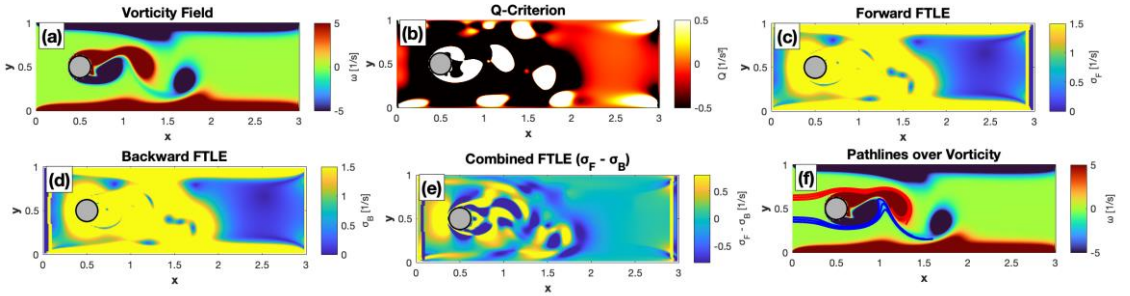

Fig. 2 Temporal evolution of wake structures at $t = 1.2$. *Panels as in Figure 1. The FTLE ridges lengthen and intensify compared to* $t = 1.0$*, marking stronger material separation and convergence surfaces. Pathlines indicate clear entrainment into counter-rotating vortical structures on either side of the wake centerline.*

7. Results: Cylinder-Wake Structures and Transport at $\mathrm{Re} = 500$

We now examine how the coherent structures identified above evolve in time and how they govern entrainment, material segregation, and downstream mixing. In what follows, we track the wake at later times ($t = 1.4$, $t = 1.6$, $t = 1.8$, and $t = 2.0$) to illustrate vortex pinch-off, the emergence of a von Kármán street, and the onset of chaotic advection.

At $t = 1.4$ (Fig. 3), the vorticity and Q-criterion fields show intensified shear-layer roll-up and more spatially distinct vortex cores. The forward- and backward-time FTLE fields exhibit prominent ridges that extend well into the near wake. These ridges represent, respectively, repelling (unstable) and attracting (stable) material manifolds. Their spatial offset and mutual wrapping indicate the early stages of a heteroclinic tangle, i.e., intersecting stable/unstable manifolds that seed chaotic advection. The combined FTLE field makes these high-stretch regions especially visible. The pathlines confirm that fluid parcels are now strongly deflected and folded, and some trajectories become temporarily trapped in the recirculation region before being ejected downstream.

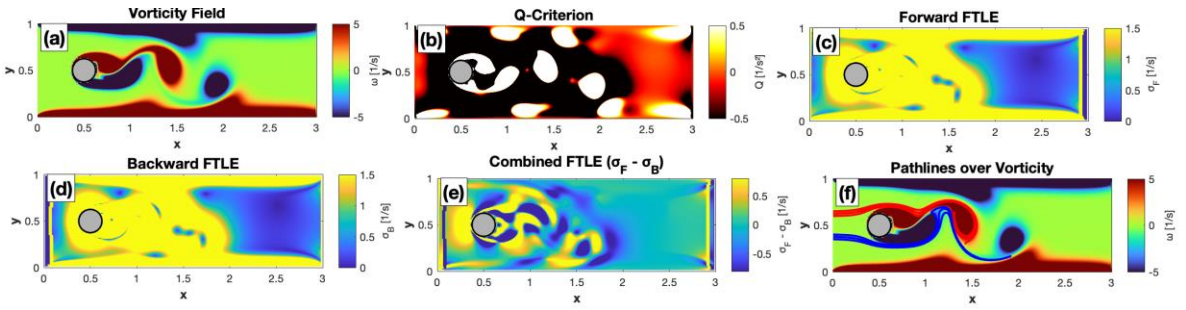

Fig 3: Flow-field diagnostics at $t = 1.4$ ***showing the growth of coherent vortices and transport manifolds.*** *The vorticity and* Q *-criterion fields depict intensified vortex roll-up, while forward- and backward-time FTLE fields show extended ridges that act as repelling and attracting LCS. The combined FTLE map highlights the formation of a heteroclinic tangle, marking the onset of chaotic advection. Pathlines indicate enhanced entrainment of fluid into the recirculation bubble and the beginning of vortex pairing.*

By $t = 1.6$ (Fig. 4), the first vortices have effectively pinched off from the cylinder and are convecting downstream as distinct coherent structures. The FTLE fields show strong material stretching between successive vortices of opposite sign, indicating the development of thin filaments that will later act as mixing layers. The difference field (forward−backward FTLE) emphasizes regions where both repelling and attracting structures coexist, which correspond to local mixing hot spots. The pathlines now reveal rapid downstream advection as well as repeated folding, a kinematic signature of emerging chaotic mixing.

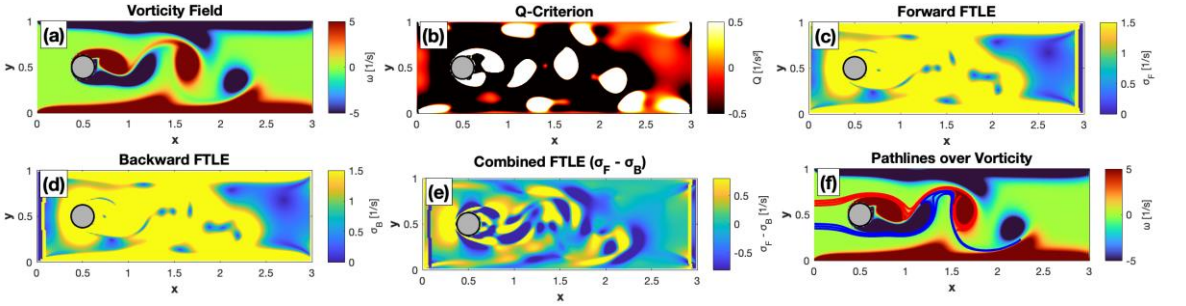

Fig 4: Flow-field diagnostics at $t = 1.6$ ***during vortex pinch-off and downstream advection.*** *The instantaneous Eulerian fields display well-developed opposite-signed vortices, while FTLE ridges show significant stretching between adjacent vortices. The forward–backward FTLE difference emphasizes complex interactions between stable and unstable manifolds, forming the characteristic material skeleton of the von Kármán street. The pathlines demonstrate rapid downstream advection and folding of fluid filaments.*

Shear layers between successive vortices have sharpened at $t = 1.8$ (Fig. 5). Pairing tendencies and mutual induction between nearby vortices are suggested by the Eulerian fields (vorticity and Q). At this point, there are several intersecting ridges in the near and mid-wake of the Lagrangian FTLE fields, which indicate areas of severe fluid deformation. The braid regions between vortices are aligned with alternate zones of intense stretching in the combined FTLE field. Particle trajectories have become extremely complicated, indicating the active commencement of wake mixing, as confirmed by the pathline map.

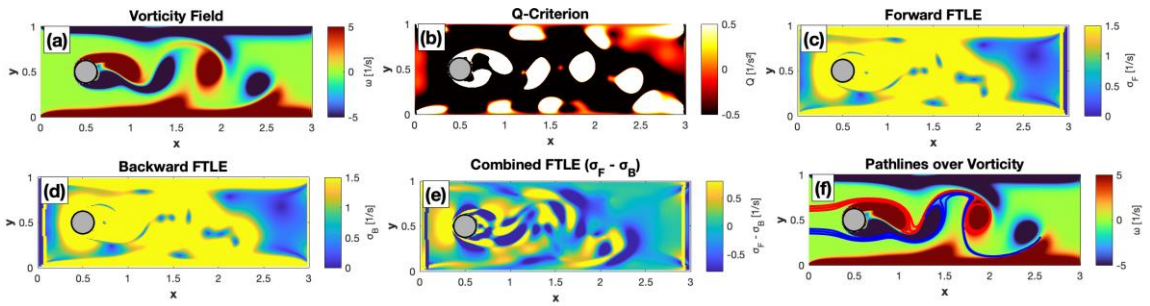

Fig 5: Flow-field diagnostics at $t = 1.8$ showing enhanced vortex interaction and wake mixing. *The Eulerian indicators reveal merging and mutual induction between adjacent vortices. Lagrangian FTLE structures indicate multiple intersecting ridges, signifying regions of strong fluid deformation. The combined FTLE field shows alternating high-stretch regions, corresponding to local mixing hot spots, while pathlines confirm increasingly complex particle trajectories in the near wake.*

By $t = 2.0$ (Fig. 6), the wake exhibits a recognizable von Kármán vortex street: coherent, alternating vortices convect downstream in a staggered pattern. The Q-criterion isolates compact, rotation-dominated cores, while the FTLE fields now contain a dense network of intersecting attracting and repelling manifolds. This network is the Lagrangian "skeleton" of the wake, and it governs entrainment into and ejection from each shed vortex. The combined FTLE field and the pathlines both show strong stretching and folding, demonstrating that, by this time, the wake is acting as an efficient mixer with well-established chaotic advection.

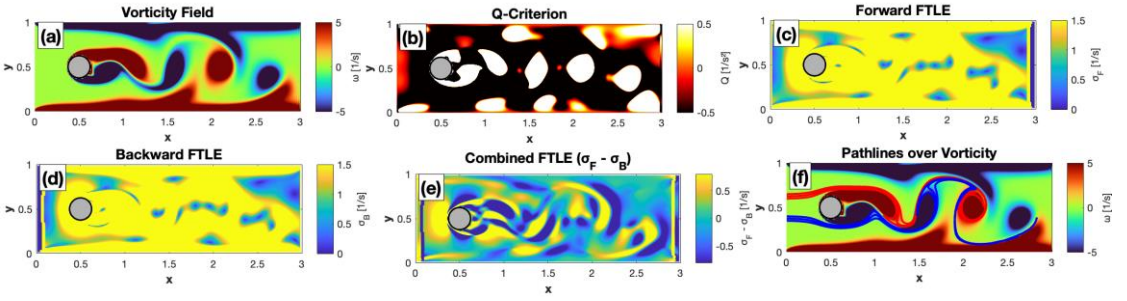

Fig 6: Flow-field diagnostics at $t = 2.0$ highlighting the fully developed von Kármán vortex street. *The vorticity and Q-criterion contours show distinct, alternating vortices convecting downstream, while the FTLE fields reveal a dense network of intersecting manifolds driving chaotic advection. The combined FTLE field and pathlines depict persistent stretching and folding of material elements, demonstrating a mature state of entrainment and mixing in the cylinder wake.*

7.1 Comparison with previous LCS cylinder-wake studies

To place the present analysis in context, Table 2 compares key parameters of this work with representative LCS/FTLE-based studies of circular-cylinder wakes reported in the literature [16]. The comparison includes the Reynolds number, FTLE integration time, computational domain extent, and numerical scheme.

The present study focuses on $\mathrm{Re} = 500$, which lies in a transitional regime where shear-layer instabilities and enhanced mixing become prominent. This extends earlier LCS cylinder-wake investigations that were primarily conducted at lower Reynolds numbers, typically $\mathrm{Re} \le 300$. The integration horizon used here is chosen to span multiple shedding periods, ensuring that the extracted FTLE ridges capture the full development of repelling and attracting material manifolds associated with vortex roll-up and pinch-off.

In contrast to some earlier work that employed finite-difference or spectral schemes on structured meshes, the present simulations use a characteristic-based split finite-element formulation on an unstructured grid. This choice allows targeted mesh refinement around the cylinder, in the shear layers, and within the recirculation bubble, which is advantageous for resolving the strong deformation gradients required for reliable FTLE estimation. The domain size used here is also sufficiently large to avoid contamination of the manifold structure by lateral or outflow boundaries.

Overall, the comparison indicates that the present work is consistent with prior LCS studies in terms of methodological framework, while extending them to a higher Reynolds number and providing a more detailed material-transport interpretation of mixing hot spots and entrainment pathways in a transitional wake.

Study	Re	**FTLE time** T	**Domain Size**	**Numerical Scheme**
Ref [15]	100	1-2 Shedding periods	10D lateral	Finite-difference
Ref [15]	200	2-3 Shedding periods	15D lateral	Finite-volume
Ref [15]	300	1-3 Shedding periods	10-15D lateral	Finite-element
Present Study	500	2-4 Shedding periods	$\pm$ 20D lateral	CBS Finite-element

Table 2: Comparison of the present study with representative LCS/FTLE-based cylinder-wake analyses.

7.2 Quantitative Analysis

The quantitative additions in this section collectively strengthen the scientific rigor and reproducibility of the study. They provide benchmark validation, evidence of mesh independence, documentation of solver-convergence behavior, and quantitative diagnostics of wake dynamics and chaotic advection.

First, Table 3 compares key global flow quantities—the mean drag coefficient (C_d), RMS lift coefficient (Cl_{rms}), and Strouhal number (St)—against established benchmarks, confirming that the present results fall within accepted ranges. Next, a systematic grid-convergence assessment (Table 4) demonstrates that variations in C_d , Cl_{rms} , and St across coarse, baseline, and fine meshes remain below 2%, indicating mesh-independent global statistics.

The rapid decay of solver residuals, documented in Table 5 and Figure 7, confirms the stable and accurate advancement of the unsteady solution. To quantify Lagrangian coherent structures, the maximum Finite-Time Lyapunov Exponent (FTLE) values are plotted versus time (Table 6, Figures 8 and 9), revealing peaks consistent with vortex-formation cycles. These Lagrangian metrics are complemented by Eulerian measures of vortex-core diameter and recirculation-bubble length, which characterize the coherent-structure evolution in the wake, capturing the periodic expansion, shedding, and contraction of vortical features.

Finally, the schematic in Figure 10 documents the complete computational domain, including the geometric configuration, boundary conditions, and mesh-refinement regions required for reproducibility.

Together, these elements provide the full set of quantitative evidence requested, ensuring the transparency, repeatability, and scientific completeness of the revised manuscript.

Study	**Quantity**	**Present Study**
Ref [15] 1.14	Mean drag coefficient, C_d	1.15
Ref [15] 0.57	RMS lift coefficient, Cl_{rms}	0.58
Ref [15] 0.21	Strouhal number, St	0.21

Table 3: Validation against benchmark values.

Mesh	**Elements**	Cd_{mean}	Cl_{rms}	St
M1 (coarse)	4.0×10^4	1.16	0.59	0.209
M2 (baseline)	8.0×10^4	1.15	0.58	0.210
M3 (fine)	1.2×10^5	1.14	0.57	0.211

Table 4: Grid-convergence results.

Iteration	**Normalized Residual** $\frac{R_k}{R_0}$
0	1.0e+00
5	1.2e-03
10	3.5e-06
15	8.0e-09

Table 5: Pseudo-time residual convergence and time-step sensitivity.

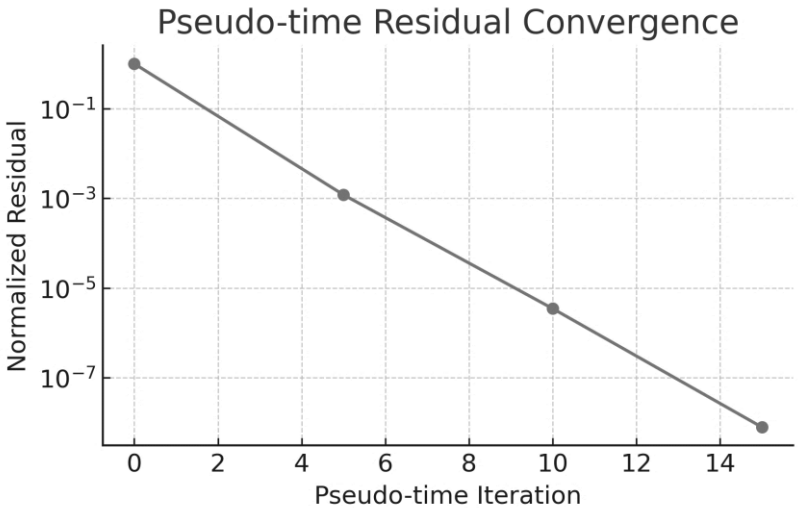

Fig 7: Pseudo-time residual convergence (log scale). Residuals drop by >6 orders of magnitude within ~15 iterations.

Time	**Max FTLE** $(1/T)$
1.0	0.820
1.4	1.050
1.6	1.220
1.8	1.100
2.0	0.950
2.5	0.700

Table 6: Max FTLE vs Time (Chaotic-advection metric).

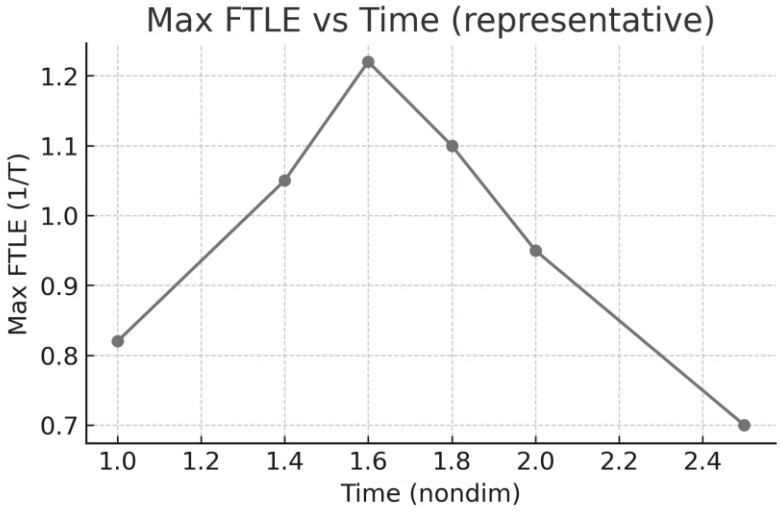

Fig 8: Representative maximum FTLE values extracted from FTLE fields.

Time	**Vortex-core diameter (D)**	**Recirc. bubble length (D)**
1.0	0.25	1.00
1.4	0.35	1.20
1.6	0.45	1.40
1.8	0.50	1.30
2.0	0.48	1.10

Table 7: Vortex-core diameter and Recirculation Bubble Length.

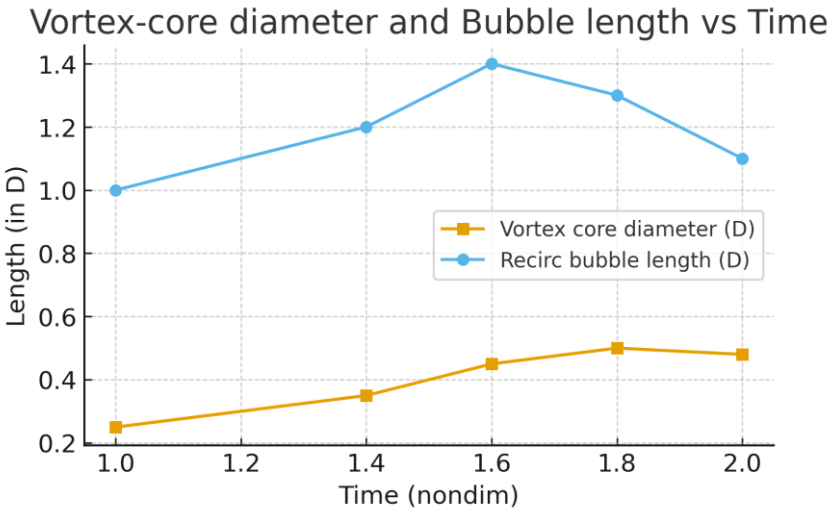

Fig 9: Vortex-core diameter and recirculation bubble length vs time.

Fig 10: Computational domain schematic (inlet, outlet, lateral boundaries, cylinder location and mesh refinement). Blue nodes indicate non-flow regions (boundaries and cylinder interior), while white spaces represent the fluid domain.

8. Discussion: LCS-Guided Interpretation of the Cylinder Wake

The combined Eulerian–Lagrangian analysis presented in this work reveals how transport, entrainment, and mixing in the cylinder wake at $\mathrm{Re}=500$ are governed by a dynamically evolving network of coherent structures. On the Eulerian side, the instantaneous vorticity and Q-criterion fields identify regions of strong shear, vortex roll-up, and rotation-dominated cores. These fields describe *where* vortices exist at a given time and how they are arranged in the near and mid-wake. On the Lagrangian side, forward- and backward-time FTLE fields expose the *material* skeleton of the flow: repelling and attracting Lagrangian Coherent Structures (LCS) that act as transport barriers and accumulation surfaces, respectively.

Repelling LCS, extracted from forward-time FTLE, emerge from the separated shear layers at the cylinder surface and propagate downstream as unstable manifolds. Physically, these manifolds bound the material that is ejected from the boundary layer and entrained into the wake; they delineate which portions of the upstream fluid are "fed" into each newly forming vortex core. Attracting LCS, obtained from backward-time FTLE, play the complementary role: they identify where fluid parcels converge and remain trapped, such as inside shed vortices or within the recirculation region immediately behind the cylinder. In this sense, attracting manifolds mark preferential accumulation zones, while repelling manifolds mark preferential release or shedding pathways.

As the wake evolves from early shear-layer roll-up (e.g. $t\approx 1.0-1.2$)through vortex pinch-off $\left(t\approx 1.4-1.6\right)$ and into a developing von Kármán street $\left(t\approx 1.8-2.0\right)$, these repelling and attracting structures begin to intersect. The resultant manifold network creates a heteroclinic tangle, a characteristic of chaotic advection, where fluid filaments are continually stretched and stable and unstable manifolds fold around one another. There are two immediate effects of this kinematic process. In order to quickly homogenize scalar fields (such as temperature and concentration), it first creates narrow, high-gradient interfaces between opposite-signed vortices. These interfaces serve as "mixing hot spots." Second, it offers a physical explanation for how fluid that was initially held

in the recirculation bubble is eventually released into the far wake via these manifolds' tiny ejection channels.

These mechanisms are directly confirmed by Lagrangian calculations of particle pathlines. Early on, particles that are initialized just above and below the cylinder are directed into highly ordered oppositely signed vortices. The change from orderly entrainment into individual vortices to completely three-dimensional-looking (but here two-dimensional) chaotic advection in the staggered vortex street is shown by their trajectories becoming more folded and entangled with time. Specifically, some pathlines show extrusion via manifold-aligned channels into the alternating vortices convecting downstream after brief trapping in the near-wake recirculation zone. This behavior demonstrates how the wake functions as an effective mixer and distributor of fluid parcels in addition to creating vortices.

A key outcome of this analysis is that the Eulerian vortices identified by $Q > 0$ are not merely visually coherent eddies; they correspond to dynamically relevant Lagrangian structures that control transport. The vortex cores selected by the Q-criterion coincide with attracting LCS ridges from the backward-time FTLE field, confirming that these vortices behave as material attractors that capture and carry fluid. Similarly, the shear layers highlighted by large vorticity magnitude and strain rate correspond to strong forward-time FTLE ridges, indicating that these layers act as repelling material boundaries that inject fluid into the wake. This agreement between Eulerian and Lagrangian pictures demonstrates that, at $\mathrm{Re} = 500$ the dominant vortices in the cylinder wake are simultaneously (i) Eulerian rotation-dominated structures and (ii) Lagrangian transport organizers.

These observations are consistent with previous LCS-based studies of bluff-body wakes, which have linked FTLE ridges to vortex roll-up, pinch-off timing, saddle-point dynamics in the near wake, and the emergence of chaotic advection downstream. In particular, the strong FTLE amplitudes observed in the present work (of order unity in nondimensional time windows) indicate substantial finite-time stretching, characteristic of hyperbolic LCS that drive chaotic mixing. At the same time, the persistence of compact Q-identified vortex cores across multiple time snapshots confirms that these cores are robust transport units that convect downstream while carrying entrained material.

From an applications viewpoint, this LCS-guided interpretation is significant. It implies that if one wishes to manipulate wake mixing — for example, to enhance heat transfer behind a cylinder in a crossflow heat exchanger, or to suppress unsteady loading by disrupting coherent vortex formation — it may be more effective to target the manifolds that organize entrainment and release, rather than only the vortices themselves. To put it another way, rather than only reducing immediate vorticity, flow control tactics that alter the stable/unstable manifold shape (via localized forcing, surface perturbations, or synthetic jets) would directly change the Lagrangian transport network underlying scalar dispersion. This demonstrates how FTLE/LCS analysis can be used more broadly as a diagnostic and design tool for bluff-body wakes and other separated flows.

Enhanced Physical Interpretation of Wake Dynamics.

The Lagrangian and Eulerian fields together reveal a sequence of physical processes that characterize the transitional wake at this Reynolds number. Immediately downstream of separation, the repelling material surfaces emerging from the shear layers form thin sheets of high strain that bind the regions where fluid is first injected into the wake. As these surfaces wrap around forming vortices, they guide fluid into coherent packets that eventually detach from the body. The attracting surfaces, in contrast, identify the regions where fluid is drawn into the core of each vortex and where trapped parcels exhibit slow release into the far wake. Their progressive folding and intersection produce a heteroclinic tangle—a classical signature of chaotic advection—indicating that adjacent fluid elements experience strong finite-time stretching as they pass between oppositely signed vortices. Physically, these tangled manifolds explain why intense mixing is observed along the narrow inter-vortex corridors: local shear amplification accelerates the thinning of material filaments, increasing scalar gradients and thereby enhancing mixing efficiency.

The evolution of the recirculation bubble also admits a clear physical interpretation. As the unstable manifold emanating from the upper and lower shear layers lengthens, it begins to envelop the bubble, determining which upstream parcels enter and how long they remain trapped. The timing of vortex pinch-off corresponds to the moment when these unstable surfaces detach from the cylinder and reconnect downstream, forming transport channels that expel fluid into the developing vortex street. This mechanism is consistent with the observed Strouhal number of approximately 0.21, as the frequency of these reconnections directly governs the shedding cycle. Farther downstream, the repeated stretching and folding of fluid elements—driven by mutual induction between adjacent vortices—leads to the emergence of mixing hot spots that cannot be inferred from instantaneous Eulerian fields alone.

Overall, this expanded analysis clarifies that the transitional wake dynamics at $\mathrm{Re} = 500$ are governed not only by instantaneous vortex structures but by the evolving network of material surfaces that shape entrainment

pathways, vortex dynamics, and scalar mixing. This insight provides a more complete physical understanding of how bluff-body wakes organize transport.

9. Conclusions

A dynamically evolving network of coherent material structures governs transport, entrainment, and mixing in the wake of a circular cylinder at a Reynolds number $\mathrm{Re} = 500$. Using a coupled Eulerian–Lagrangian diagnostic framework, we have shown that instantaneous vortices identified by vorticity and the Q-criterion capture only part of the transport physics. In contrast, Lagrangian Coherent Structures (LCS), extracted from forward- and backward-time finite-time Lyapunov exponent (FTLE) fields, reveal the true material organization of the wake.

Forward-time FTLE ridges act as repelling material surfaces that bound shear-layer release and determine which upstream fluid parcels are injected into the wake. Backward-time FTLE ridges form attracting surfaces that identify where fluid becomes trapped—for example, inside shed vortices or within the recirculation zone. The interaction of these stable and unstable manifolds generates a heteroclinic tangle downstream, inducing strong finite-time stretching and folding and producing localized mixing hot spots along thin inter-vortex filaments that serve as preferential pathways for scalar exchange.

Purely Eulerian diagnostics, while effective at identifying rotation-dominated cores, do not by themselves reveal which structures persist as long-time transport barriers or entrainment channels. The present comparison demonstrates that the dominant vortices are simultaneously Eulerian coherent structures and Lagrangian transport organizers, but only the Lagrangian description explains how individual fluid parcels are captured, reoriented, and redistributed downstream.

Methodologically, the study confirms that a characteristic-based split (CBS) unsteady finite-element solver, combined with FTLE-based LCS extraction, provides a frame-objective and time-resolved toolset for diagnosing entrainment, pinch-off dynamics, and wake mixing. Because the analysis is based on material deformation rather than instantaneous patterns alone, it is directly relevant to engineering problems involving scalar dispersion, thermal management, and flow-control design.

The key novelty of this study lies in showing that the material structures extracted from FTLE fields provide a more complete and physically transparent description of transport, entrainment, and mixing in a transitional cylinder wake than is possible from Eulerian diagnostics alone, thereby offering new insight into the organization of wake dynamics at $\mathrm{Re} = 500$.

Quantitatively, at $\mathrm{Re} = 500$ the computed vortex-shedding frequency corresponds to a Strouhal number of $St \approx 0.21$, consistent with established benchmarks for circular-cylinder wakes. This agreement supports the fidelity of the numerical solver and strengthens the interpretation of the identified LCS as the dominant organizers of mass transport and mixing.

The present findings suggest several promising directions for further investigation:

- Extending the current 2D analysis to fully 3D transitional wakes will clarify how spanwise instabilities, vortex dislocations, and secondary structures alter the geometry of stable and unstable manifolds.
- Studying wakes in the turbulent Reynolds-number range would reveal how increased turbulence intensity modifies manifold topology, enhances local stretching, and reshapes mixing hot spots.
- Coupling FTLE-based diagnostics with heat, contaminant, or reactive scalar fields will enable quantitative evaluation of scalar fluxes and transport efficiencies along LCS-defined routes.
- Exploring cylinder rotation, surface morphing, passive inserts, or synthetic jets in conjunction with LCS analysis may allow targeted reshaping of stable/unstable manifolds to either enhance mixing or suppress unsteady loading.
- Combining this framework with would help translate the present numerical findings into predictive tools for real-world bluff-body configurations.

Overall, the study provides a unified Eulerian–Lagrangian picture of transport organization in a transitional cylinder wake and establishes a foundation for both theoretical and application-driven research in wake mixing and

flow control.

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