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RESEARCH PAPER

A Fluid Dynamics-based Modified Murray Law for Hierarchical Vein Networks in Lotus Leaves: Geometry, Transport Mechanism and Biological Implications

Yan-Ping Liu [a], Ji-Huan He [b,c, d, e,*]

[a] *College of Petrochemical Engineering and Environment, Zhejiang Ocean University, Zhoushan, 316022, Zhejiang, China*

[b] *School of Jia Yang; Zhejiang Shuren University, Hangzhou; Zhejiang, China*

[c] *School of Information Engineering, Yango University, Fuzhou 350015, China*

[d] *School of Mathematics and Big Data, Hohhot Minzu College, Hohhot, Inner Mongolia 010051, China*

[e] *Department of Mathematical Sciences, Saveetha School of Engineering, SIMATS, Chennai, Tamil Nadu, India*

Abstract

Lotus leaf is famous for its so-called lotus effect, which has evoked many advanced biomimetic designs, especially for surface's wetting properties, its tree-like veins are also a source of brilliant innovations. The geometry of the veins is similar to that of blood vessels, where Murray's law can elucidate the branched structure. However, the law is not valid for the lotus' vein structure. Here we find a new law to reveal the geometry, and its biological understanding is elucidated, furthermore Murray's law is a special case of the new found law. Our findings may be of great biological and technological importance, especially in plant ecology, urban traffic planning, watershed planning, and chemical engineering.

Keywords: Hierarchical structure; Fluid mechanics; Surface-enhanced transport; Modified Murray law; Geometric potential.

1. Introduction

The lotus leaf has been a subject of scientific study for a considerable period, distinguished by its distinctive "lotus effect," a phenomenon characterized by its exceptional superhydrophobicity and self-cleaning behavior, attributable to its nano/micro-scale surface morphology [1]. This distinctive property has led to the development of pioneering biomimetic designs, encompassing self-cleaning coatings [2, 3], omniphobic surfaces [4], and microelectromechanical systems (MEMS) [5, 6]. The periodic solutions of MEMS systems have undergone extensive research [7, 8]. The geometrical potential theory [9] has provided theoretical frameworks for explaining the wetting properties, while empirical studies have demonstrated its mechanical robustness [10].

The hierarchical structure of the lotus leaf has been employed for a variety of purposes, including vibration-proofing [11, 12], the creation of artificial blood vessels [13], the development of soft machines [14, 15], and the fabrication of electronic skins [16].

Despite the extensive research conducted on the leaf's surface architecture, the hierarchical vein structure—with its tree-like branching morphology—has received comparatively less attention, notwithstanding its critical role in fluid transport and mechanical support. Early anatomical studies of leaf veins primarily focused on taxonomic implications. In contrast, recent investigations have highlighted the functional significance of vein networks in plant ecology, paleoenvironmental reconstruction, and urban infrastructure planning (e.g., traffic and watershed

* Corresponding author. Tel.: +86-139-1315-2427

E-mail address: hejihuan@ygu.edu.cn, hejihuan@suda.edu.cn

management).

The branching pattern of lotus leaf veins exhibits a remarkable similarity to biological vascular systems, thus prompting investigations into the applicability of Murray's law—a seminal principle in fluid dynamics that describes optimal branching in transport networks [17]. Murray's law postulates that the cube of the mother vessel's diameter is equivalent to the sum of the cubes of the daughter vessel diameters. This relationship is extensively employed in the context of modeling blood vessels [18] and microfluidic devices [19, 20]. However, our preliminary experiments reveal significant discrepancies between Murray's law predictions and measured vein diameters, indicating its inadequacy for describing lotus leaf venation. This invalidity is consistent with recent findings in non-vascular systems, such as the lymphatic network [21], highlighting the necessity for a revised theoretical framework.

The objective of this study is to establish a novel law for lotus leaf vein geometry, integrating fluid mechanics and geometric potential theory. It is demonstrated that the novel legislation not only rectifies the limitations of Murray's law but also incorporates it as a particular case. The proposed framework underscores the pivotal function of surface-enhanced transport, a phenomenon propelled by geometric potential and surface tension. This framework elucidates its ramifications for biological fluid dynamics and engineering applications, encompassing microchannel design, ecological modeling, and urban infrastructure optimization.

2. Lotus leaf's vein geometry and fluid transport framework

The lotus leaf exhibits a distinctive hierarchical venation system, see Fig.1, characterized by a central primary vein branching into secondary and tertiary veins in a tree-like architecture. This morphology, as visualized in Fig. 1, resembles biological vascular networks in both geometry and branching patterns, with each vein tree featuring bifurcated structures that distribute fluids across the leaf surface. Microscopic observations reveal that the vein diameter decreases exponentially with branching order, while the bifurcation angles vary dynamically to adapt to spatial constraints. Notably, 20 vein trees were initially identified on the measured leaf surface, though four were excluded due to abnormal development or severe geometric distortion, highlighting the influence of environmental factors on venation morphology.

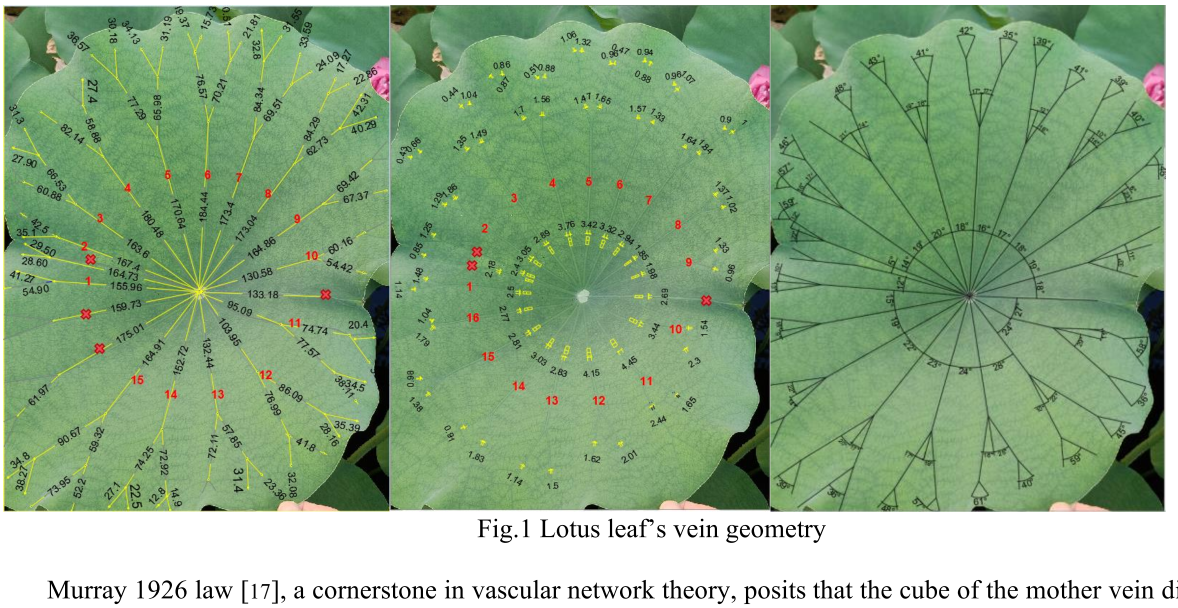

Fig.1 Lotus leaf's vein geometry

Murray 1926 law [17], a cornerstone in vascular network theory, posits that the cube of the mother vein diameter (d_0) equals the sum of the cubes of daughter veins (d_1 and d_2):

$$(d_0)^3 = (d_1)^3 + (d_2)^3 \tag{1}$$

Eq.(1) is a widely adopted analytical tool in the study of diverse biological networks [18], as well as a key design principle to design hierarchical structures [23,24]. For sample No. 6 (d_0 = 3.32mm, d_1= 1.57 mm, d_2= 1.33mm), calculations yield $3.32^3 = 36.59$ vs. $1.57^3 + 1.33^3 = 6.2225$, demonstrating a 83% discrepancy. This invalidity is consistent across 16 samples, see Fig. 2, where the predicted $(d_0)^3$ by Murray's law systematically overestimates experimental data.

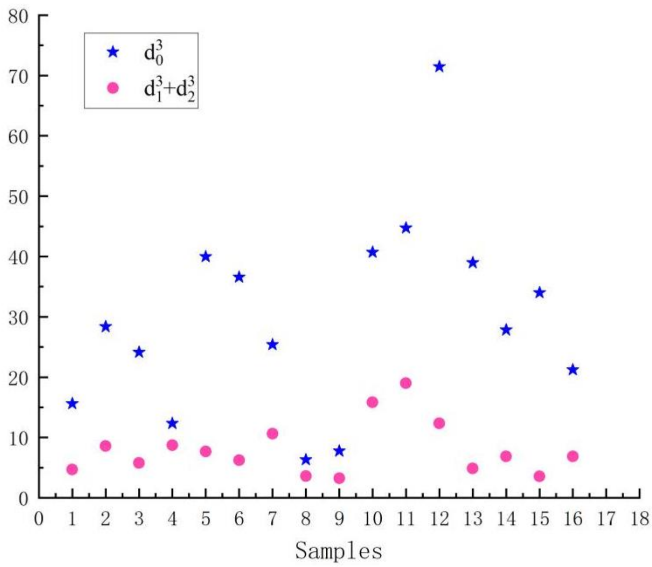

Fig.2 Prediction by Murray's law (star points) vs the experimental data (cycle points)

Previous modifications of Murray's law also fail to describe lotus venation. Talkington et al. [21] proposed an exponent of 1.45 for lymphatic vessels:

$$(d_0)^{1.45} = (d_1)^{1.45} + (d_2)^{1.45} \tag{2}$$

Yet for sample No. 6, $3.32^{1.45}$= 5.69 vs. $1.57^{1.45}+1.33^{1.45}$ = 3.43, the 40% error suggests the potential invalidity of the modified law in this study. Tree-like branching networks have been employed extensively in the study of permeability in porous concrete [22, 23], microchannels [24-26], and natural fibers [27]. In a study by Li and Yu [28], a modified Murray law was proposed as a means of optimizing thermal conductivity, which is

$$(d_0)^2 = (d_1)^2 + (d_2)^2 \tag{3}$$

But 3.32^2= 11.02 vs. $1.57^2+1.33^2$= 4.23, the 62% error shows that Eq. (3) is invalid for the lotus surface's vein trees. Some researchers have employed fractal theory to examine the Murray law [29]. The tree-like geometry suggests a fractal modification as a natural consequence, which can be written in the form [29]:

$$(d_0)^\alpha = (d_1)^\alpha + (d_2)^\alpha \tag{4}$$

where α is the fractal dimensions. The tree-like vein is of fractal character, and the two-scale fractal geometry [30-32] can model its geometry, and the fractional models can model the diffusion in the network [33-36]. It should be pointed out that the vein tree is something like a fractal partner, but Eq. (4) might be valid for some special cases. Nevertheless, it should be noted that the value of α in the aforementioned equation is not equivalent to the fractal dimensions, so fractal modifications [29] also prove inadequate, indicating that lotus veins require a fundamentally new theoretical framework.

For laminar flow in cylindrical veins, the Hagen-Poiseuille equation describes volumetric flow rate

$$\mathrm{Q} = \frac{\pi r^4 \Delta P}{8\mu L} \tag{5}$$

where r is the vein radius, ΔP is the pressure gradient, μ is fluid viscosity, and L is vein length. This equation highlights that flow rate scales with the fourth power of diameter, underscoring diameter's critical role in fluid transport. For lotus veins, the non-Newtonian properties of phloem sap (e.g., viscosity $\mu \approx 10^{-3}$ Pa·s) and varying L necessitate integrated modeling of branching networks.

Based on 16 valid vein trees, we derive the following relationships for diameter and length:

$$d_0 = \frac{d_1}{\cos\alpha_1} + \frac{d_2}{\cos\alpha_2} \tag{6}$$

$$L_0 = \frac{L_1}{\cos\alpha_1} + \frac{L_2}{\cos\alpha_2} \tag{7}$$

where d_0 and L_0 represent the diameter and length of the mother vein, while di and L_i (i=1,2) correspond to the diameters and lengths of the daughter veins. A total of 16 primary vein trees were selected on a lotus surface, and the resulting comparison is presented in Figs. 3 and 4, respectively. The data indicate a relatively high degree of agreement. The sample No.4 has the largest error, the second cascade has not bifurcated, see Fig.1.

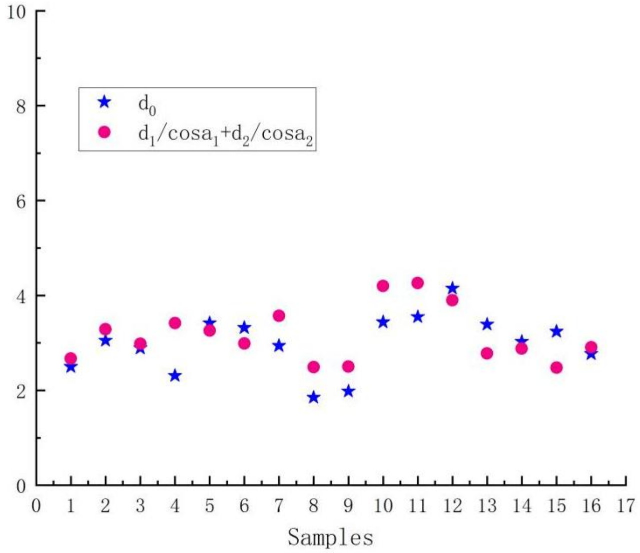

Fig.3 Comparison of the vein's diameter by our model and the experimental data

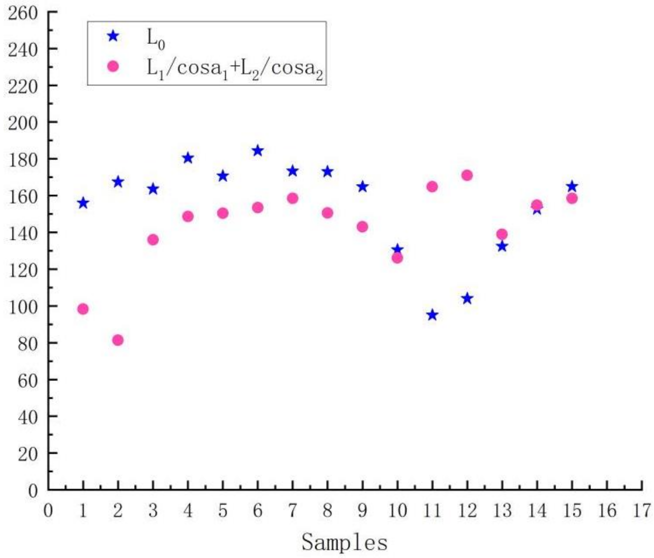

Fig.4 Comparison of the vein's length by our model and the experimental data

The accuracy of the prediction regarding the length of the daughter veins has been found to be relatively poor. This is due to the fact that the morphology of each vein tree within the lotus leaf is contingent upon the external and internal environmental conditions. To illustrate, the angle between the two adjacent trees marked with "X" in Fig. 1 is only 5 degrees, a value that is considerably less than the others. The intense competition for space results in the two trees exhibiting a pronounced degree of curvature.

The geometric pattern expresses a simple aesthetic with a simple mathematical formulation, the simpler is the better, it can also be seen as the most efficient representation of mass, energy or electricity transport on the divine nature, which is based on Plato's ideas that God created the universe according to a geometric plan— "God geometrizes continually".

Combining with Hagen-Poiseuille flow, the flow distribution ratio between daughter veins is:

$$\frac{Q_1}{Q_2} = (\frac{d_1}{d_2})^4 \frac{\cos\alpha_2}{\cos\alpha_1} \tag{8}$$

This reveals that flow allocation depends on both diameter and branching angle, explaining why Murray's law—ignoring angle effects—fails for lotus veins.

3. Blood vessels vs plant veins: Fluid transport contrast

A fundamental distinction between biological vascular systems lies in the vectorial direction of fluid transport. In mammalian blood vessels, blood flows from the high-pressure aorta to low-pressure capillaries, a unidirectional process optimized for oxygen and nutrient delivery. By contrast, lotus leaf veins facilitate transport in the opposite direction: fluids (e.g., phloem sap) move from minute terminal veins toward the primary vein, driven by osmotic pressure and transpiration pull (Fig. 5).

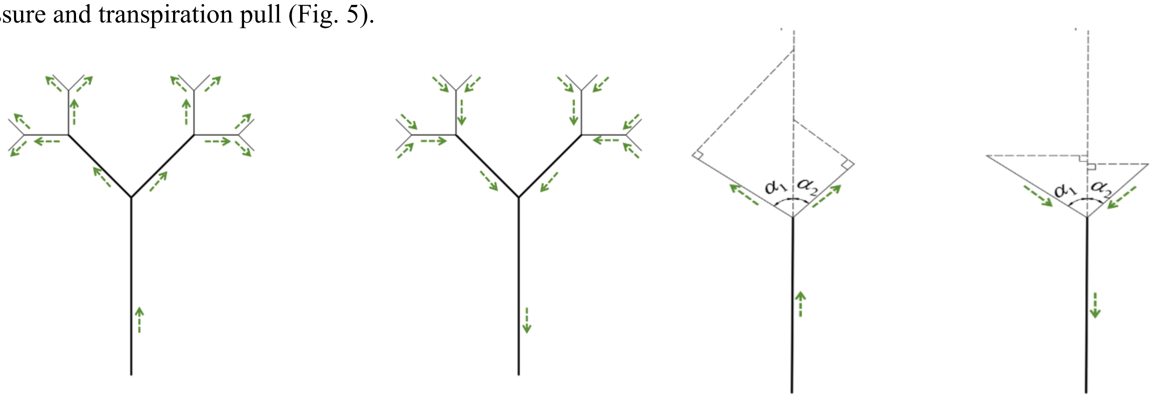

Fig. 5 Blood vessels vs plant veins

This inverse transport vector has profound implications for network geometry, as evidenced by the following contrasts:

1. The concept of branching polarity is a critical component in this field of study. Blood vessels exhibit "downstream" branching, also known as "mother-to-daughter" branching, while lotus veins feature "upstream" branching, otherwise referred to as "daughter-to-mother" branching. This results in divergent versus convergent flow patterns.

2. The process of diameter scaling is a critical component in the overall analysis. In the context of blood vessels, it has been observed that the diameter undergoes a decrease in proportion to the branching order, with the aorta exhibiting a lower diameter compared to the capillaries. Conversely, lotus veins demonstrate an increase in diameter as they approach the primary vein. This design, characterized by an increase in diameter towards the primary vein, serves to minimize flow resistance in the context of convergent transport.

3. The pressure gradient is defined as the difference in pressure between two points, which is calculated by dividing the difference in pressure by the distance between the two points. It is important to note that blood vessels rely on cardiac-driven pressure gradients, while plant veins depend on osmotic and surface tension-driven potentials. This highlights the existence of distinct energy sources for fluid motion.

The inverse transport direction necessitates a modified theoretical framework for plant veins. For blood vessels, Murray's law assumes downstream flow, but for lotus veins, we derive the following relationships by incorporating transport vector and branching angles:

$$d_0 \cos\alpha_0 = d_1 \cos\alpha_1 + d_2 \cos\alpha_2 \tag{9}$$

$$L_0 \cos\alpha_0 = L_1 \cos\alpha_1 + L_2 \cos\alpha_2 \tag{10}$$

where α_0 is the angle of the mother vein relative to the transport vector, and α_1, α_2 are daughter vein angles. This formulation accounts for the vectorial component of flow, as each branch's contribution is weighted by its cosine angle to the main transport direction.

In the event that $L_1 = L_2$, $d_1 = d_2$ and $\alpha_1 = \alpha_2$, the following is true:

$$L_0 = 2L_1 \cos\alpha_1 \tag{11}$$

$$d_0 = 2d_1 \cos\alpha_1 \tag{12}$$

When $\alpha_1 = \alpha_2 = \pi / 4$, we have

$$d_0 = 2d_1 \cos(\frac{\pi}{4}) = \sqrt{2} d_1 \tag{13}$$

This is the Li-Yu law [28], which describes the optimal thermal conduction process. When $\alpha_1 = \alpha_2 = 0.8893$, the result is

$$d_0 = 2d_1 \cos(0.8892939) = 0.6299 \tag{14}$$

This is a requisite consequence of Murray's law [17]:

$$d_0 = 2^{1/3} d_1 = 0.6299 d_1 \tag{15}$$

Consequently, our model, as delineated in Eqs. (11) and (12), encompasses both Li-Yu law and Murray's law as particular instances.

4. Surface-enhanced fluid transport mechanism

The preceding theory posited that the branching network should maximize hydraulic conductance [37]. Our experiment demonstrates, however, that it is vector transport through the surface, with the surface of the veins playing an important role in the transport process. To ascertain how this optimal outcome may be attained, we put forth a novel concept: the surface-enhanced transport. All previous theories assumed that transport occurred through microtubules. The results in this paper demonstrate that the optimal network is achieved by the surface. It is therefore evident that the surface area of the network should be maximized in order to facilitate the most efficient reception and distribution of substances. The process of transport itself is of lesser importance in comparison to the aforementioned reception and distribution. In the event that the quantity of nutrients is insufficient, the speed of transport is of no consequence.

The lotus vein's nano/micro-scale topography generates a significant surface tension gradient, driving fluid transport via the Young-Laplace equation:

$$\Delta P = \gamma(\frac{1}{R_1} + \frac{1}{R_2}) \tag{16}$$

where ΔP is the pressure difference across the fluid interface, γ is the surface tension coefficient, and R_1, R_2 are the principal radii of curvature. For lotus veins, the hierarchical roughness (e.g., micro-papillae and nano-wax crystals) creates curved interfaces with $R_1 < 100$ nm, $R_2 < 100$ nm, yielding $\Delta P \approx 10^4$ Pa—sufficient to drive fluid motion independent of osmotic pressure.

This surface tension effect is formalized by the geometric potential theory [9], which defines the potential energy per unit area as:

$$\Phi = \gamma(1+\frac{2h}{R}) \tag{17}$$

where h is the feature height of surface roughness. For lotus leaves, $h \simeq 5 \sim 10\mu m$ and $R \simeq 50 \sim 100nm$, giving $\Phi \simeq 10^2 J/m^2$—an order of magnitude higher than smooth surfaces.

Combining Young-Laplace and Hagen-Poiseuille theories, the fluid velocity (v) in lotus veins is governed by:

$$v \propto \sqrt{\frac{\gamma\Phi}{\mu L}} \tag{18}$$

This relationship reveals that surface-enhanced transport scales with the square root of geometric potential, explaining why nano-structured veins facilitate up to 100,000-fold faster flow than predicted by traditional fluid mechanics [38].

We consider a special case when $\alpha_1 = \alpha_2 = \pi/3$, this results in

$$d_0 = 2d_1\cos(\frac{\pi}{3}) = d_1 \tag{19}$$

$$L_0 = 2L_1\cos(\frac{\pi}{3}) = L_1 \tag{20}$$

That means $L_0 = L_1 = L_2$ and $d_0 = d_1 = d_2$, such a case arises in the minimum tree problem [39], and it is the fundamental geometry pattern in carbon nanotubes and graphene [40, 41]. This geometrical structure provides a rapid surface-enhanced transport of thermal energy [42] and surface-enhanced transport of electrical current [43, 44], which gives rise to a multitude of applications in engineering [45, 46].

5. Discussion

In this article, we present a new law for the geometry of veins on the lotus surface. The formulations given in Eqs. (6) and (7) and the modified Murray law given in Eqs. (9) and (10) suggest the existence of a hidden surface-enhanced transport phenomenon. This phenomenon is characterized by a dependence of transport efficiency on the surface of the branches and the surface energy (geometrical potential) [9, 47], which may serve as the primary factor for the transfer of water, nutrients, and metabolites. This finding has significant potential for technological advancement in a number of fields, including plant ecology, global warming index, urban traffic planning, watershed planning, drug delivery, and others. The surface-enhanced tree-like network also offers promising applications in permeability, diffusion, catalysis, electrode materials, and reactors. The implications of this research are far-reaching, with potential applications in biomimetic optimized networks for water evaporation [48], fractal microreactors [49], and high filtration efficiency [50]. It is our expectation that this assay will serve as a foundation for more advanced investigations into the tree-like branched structure, with a view to potential applications in both biological and technological contexts.

The following limitations are currently in effect:

1. The phenomenon of species specificity is defined as follows: The model has been validated in lotus leaves; however, further testing in other plant species (e.g., ferns, succulents) is necessary to confirm its universality.

2. The occurrence of non-Newtonian effects has been observed. The viscosity of phloem sap was found to vary, and future studies could incorporate the phenomenon of shear-thinning properties.

3. The dynamic environmental factors are as follows: Spatial competition between vein trees (e.g., 5° angle cases) was observed; however, the long-term environmental stress effects (e.g., drought, temperature) remain uninvestigated.

Future research may explore the following:

Multi-physics modeling of vein transport is a multifaceted approach that integrates various physical phenomena, including osmotic pressure, surface tension, and transpiration pull.

The employment of 3D bioprinting technology to engineer synthetic vein networks has emerged as a novel approach for conducting in vitro transport studies.

The utilization of renewable energy has been demonstrated in a variety of applications, including lotus-inspired solar water evaporation systems.

6. Conclusion

This study proposes a novel theoretical framework for understanding biological transport networks. Integration of fluid mechanics, geometric potential theory, and experimental validation is demonstrated to reveal that lotus leaf veins exhibit a modified Murray law, optimized for surface-enhanced transport. The new framework:

1. This study resolves the invalidity of traditional vascular models in plant systems.
2. It unifies diverse branching laws under a single theoretical umbrella.
3. It facilitates the development of biomimetic designs, with applications ranging from microfluidics to energy and urban planning.

The findings emphasize the significance of interdisciplinary research at the intersection of biology, mechanics, and materials science, thereby establishing a foundation for the development of next-generation technologies that draw inspiration from natural transport networks.

References

[1] A. Marmur, The Lotus Effect: Superhydrophobicity and Metastability, *Langmuir,* Vol. 20, No. 9, pp. 3517-3519, 2004/04/01, 2004.

[2] C. Neinhuis, W. Barthlott, Characterization and Distribution of Water-repellent, Self-cleaning Plant Surfaces, *Annals of Botany,* Vol. 79, No. 6, pp. 667-677, 1997/06/01/, 1997.

[3] B. Bhushan, Y. C. Jung, Natural and biomimetic artificial surfaces for superhydrophobicity, self-cleaning, low adhesion, and drag reduction, *Progress in Materials Science,* Vol. 56, No. 1, pp. 1-108, 2011/01/01/, 2011.

[4] T.-S. Wong, S. H. Kang, S. K. Y. Tang, E. J. Smythe, B. D. Hatton, A. Grinthal, J. Aizenberg, Bioinspired self-repairing slippery surfaces with pressure-stable omniphobicity, *Nature,* Vol. 477, No. 7365, pp. 443-447, 2011/09/01, 2011.

[5] S. Achanta, J.-P. Celis, *Nanotribology of MEMS/NEMS*, in: E. Gnecco, E. Meyer, *Fundamentals of Friction and Wear on the Nanoscale*, Eds., pp. 631-656, Cham: Springer International Publishing, 2015.

[6] E. Flinn, Lotus leaf yields slick idea for MEMS, *Aerosp. Am,* Vol. 5, pp. 24-5, 2005.

[7] J.-H. He, PERIODIC SOLUTION OF A MICRO-ELECTROMECHANICAL SYSTEM, *2024*, pp. 12, 2024-07-31, 2024.

[8] J.-H. He, Q. Bai, Y.-C. Luo, D. Kuangaliyeva, G. Ellis, Y. Yessetov, P. Skrzypacz, Modeling and numerical analysis for MEMS graphene resonator, *Frontiers in Physics,* Vol. Volume 13 - 2025, 2025-April-25, 2025. English

[9] D. Tian, X.-X. Li, J.-H. He, Geometrical potential and nanofiber membrane's highly selective adsorption property, *Adsorption Science & Technology,* Vol. 37, No. 5-6, pp. 367-388, 2019.

[10] Y. Zhang, H. Wu, X. Yu, F. Chen, J. Wu, Microscopic Observations of the Lotus Leaf for Explaining the Outstanding Mechanical Properties, *Journal of Bionic Engineering,* Vol. 9, No. 1, pp. 84-90, 2012/03/01, 2012.

[11] Q. He, Y. Wang, H. Gu, J. Feng, H. Zhou, Dynamic crushing analysis of a circular honeycomb with leaf vein branched characteristic, *Mechanics of Materials,* Vol. 153, pp. 103566, 2021/02/01/, 2021.

[12] Q. He, Y. Wang, X. Shi, X. Jing, Y. Jiang, Crushing behavior on the cylindrical tube based on lotus leaf vein branched structure, *Mechanics of Materials,* Vol. 165, pp. 104205, 2022/02/01/, 2022.

[13] K. Guo, M. Liu, D. Vella, S. Suresh, K. J. Hsia, Dehydration-induced corrugated folding in <i>Rhapis excelsa</i> plant leaves, *Proceedings of the National Academy of Sciences,* Vol. 121, No. 17, pp. e2320259121, 2024.

[14] M. Tian, J. Shuai, B. A. Bishop, W. Zhang, J. Chen, X. Wang, Plant cellulose-based biomimetic artificial Small-Diameter vascular materials enabled by gradient Dual-Network entanglement, *Chemical Engineering Journal,* Vol. 476, pp. 146751, 2023/11/15/, 2023.

[15] W. Wu, R. M. Guijt, Y. E. Silina, M. Koch, A. Manz, Plant leaves as templates for soft lithography, *RSC Advances,* Vol. 6, No. 27, pp. 22469-22475, 2016.

[16] B. Feng, T. Sun, W. Wang, Y. Xiao, J. Huo, Z. Deng, G. Bian, Y. Wu, G. Zou, W. Wang, T. Ren, L. Liu, Venation-Mimicking, Ultrastretchable, Room-Temperature-Attachable Metal Tapes for Integrated Electronic Skins, *Advanced Materials,* Vol. 35, No. 8, pp. 2208568, 2023.

[17] C. D. Murray, The Physiological Principle of Minimum Work, *Proceedings of the National Academy of Sciences,* Vol. 12, No. 3, pp. 207-214, 1926.

[18] J. Haskovec, P. Markowich, G. Pilli, Murray's law for discrete and continuum models of biological networks, *Mathematical Models and Methods in Applied Sciences,* Vol. 29, No. 12, pp. 2359-2376, 2019.

[19] D. Jing, S. Song, Y. Pan, X. Wang, Optimal fractal tree-like microchannel networks with slip for laminar-flow-modified Murray's law, *Beilstein Journal of Nanotechnology,* Vol. 9, pp. 482-489, //, 2018.

[20] C. Hou, L. Li, R. Lv, Z. Tian, X. Chen, Mixing Performance of Micromixers with Fractal Obstacles Based on Murray's Law, *Chemical Engineering & Technology,* Vol. 44, No. 12, pp. 2220-2227, 2021.

[21] A. M. Talkington, R. B. Davis, N. C. Datto, E. R. Goodwin, L. A. Miller, K. M. Caron, Dermal Lymphatic Capillaries Do Not Obey Murray's Law, *Frontiers in Cardiovascular Medicine,* Vol. Volume 9 - 2022, 2022-April-12, 2022. English

[22] C.-H. He, C. Liu, FRACTAL DIMENSIONS OF A POROUS CONCRETE AND ITS EFFECT ON THE CONCRETE'S STRENGTH, *2023*, pp. 14, 2023-04-10, 2023.

[23] C.-H. He, H.-W. Liu, C. Liu, A FRACTAL-BASED APPROACH TO THE MECHANICAL PROPERTIES OF RECYCLED AGGREGATE CONCRETES, *2024*, pp. 14, 2024-07-31, 2024.

[24] T. Miao, A. Chen, L. Zhang, B. Yu, A novel fractal model for permeability of damaged tree-like branching networks, *International Journal of Heat and Mass Transfer,* Vol. 127, pp. 278-285, 2018/12/01/, 2018.

[25] S. YANG, H. FU, B. YU, FRACTAL ANALYSIS OF FLOW RESISTANCE IN TREE-LIKE BRANCHING NETWORKS WITH ROUGHENED MICROCHANNELS, *Fractals,* Vol. 25, No. 01, pp. 1750008, 2017.

[26] T. Miao, A. Chen, Y. Xu, S. Yang, B. Yu, Optimal structure of damaged tree-like branching networks for the equivalent thermal conductivity, *International Journal of Thermal Sciences,* Vol. 102, pp. 89-99, 2016/04/01/, 2016.

[27] J. Fan, X. Yang, Y. Liu, Fractal calculus for analysis of wool fiber: Mathematical insight of its biomechanism, *Journal of Engineered Fibers and Fabrics,* Vol. 14, pp. 1558925019872200, 2019.

[28] L. Li, B. Yu, Fractal analysis of the effective thermal conductivity of biological media embedded with randomly distributed vascular trees, *International Journal of Heat and Mass Transfer,* Vol. 67, pp. 74-80, 2013/12/01/, 2013.

[29] B. Zhou, Q. Cheng, Z. Chen, Z. Chen, D. Liang, E. A. Munro, G. Yun, Y. Kawai, J. Chen, T. Bhowmick, K. K. Padmanathan, L. G. Occhipinti, H. Matsumoto, J. W. Gardner, B.-L. Su, T. Hasan, Universal Murray's law for optimised fluid transport in synthetic structures, *Nature Communications,* Vol. 15, No. 1, pp. 3652, 2024/05/07, 2024.

[30] G. Feng, A CIRCULAR SECTOR VIBRATION SYSTEM IN A POROUS MEDIUM, *Facta Universitatis, Series: Mechanical Engineering*, 2023.

[31] Y.-P. LIU, C.-H. HE, K. A. GEPREEL, J.-H. HE, CLOVER-INSPIRED FRACTAL ARCHITECTURES: INNOVATIONS IN FLEXIBLE FOLDING SKINS FOR SUSTAINABLE BUILDINGS, *Fractals,* Vol. 0, No. 0, pp. 2550041.

[32] Y.-P. Liu, J.-H. He, M. H. Mahmud, Leveraging Lotus Seedsâ€™ Distribution Patterns For Fractal Super-Rope Optimization, *FRACTALS (fractals),* Vol. 33, No. 03, pp. 1-11, 2025.

[33] Y. Cheng, C.-L. Luo, C. Zhong, H. Lin, D. Marinkovic, J.-H. He, Differential equation-driven intelligent control: Integrating AI, Quantum computing, and adaptive strategies for next-generation industrial automation, *Advances in Differential Equations and Control Processes,* Vol. 32, No. 1, pp. 3096, 04/24, 2025.

[34] S. Zhang, H. Zhang, Y. Wang, Z. Li, Dynamic properties and numerical simulations of a fractional phytoplankton-zooplankton ecological model, *Networks and Heterogeneous Media,* Vol. 20, No. 2, pp. 648-669, 2025.

[35] C. Han, Y.-L. Wang, Z.-Y. Li, A high-precision numerical approach to solving space fractional Gray-Scott model, *Applied Mathematics Letters,* Vol. 125, pp. 107759, 2022/03/01/, 2022.

[36] H. Che, Y.-L. Wang, Z.-Y. Li, Novel patterns in a class of fractional reaction–diffusion models with the Riesz fractional derivative, *Mathematics and Computers in Simulation,* Vol. 202, pp. 149-163, 2022/12/01/, 2022.

[37] K. A. McCulloh, J. S. Sperry, F. R. Adler, Water transport in plants obeys Murray's law, *Nature,* Vol. 421, No. 6926, pp. 939-942, 2003/02/01, 2003.

[38] M. Majumder, N. Chopra, R. Andrews, B. J. Hinds, Enhanced flow in carbon nanotubes, *Nature,* Vol. 438, No. 7064, pp. 44-44, 2005/11/01, 2005.

[39] Y. Zhang, Y.-W. Tan, H. L. Stormer, P. Kim, Experimental observation of the quantum Hall effect and Berry's phase in graphene, *Nature,* Vol. 438, No. 7065, pp. 201-204, 2005/11/01, 2005.

[40] S. Stankovich, D. A. Dikin, G. H. B. Dommett, K. M. Kohlhaas, E. J. Zimney, E. A. Stach, R. D. Piner, S. T. Nguyen, R. S. Ruoff, Graphene-based composite materials, *Nature,* Vol. 442, No. 7100, pp. 282-286, 2006/07/01, 2006.

[41] S. Iijima, T. Ichihashi, Single-shell carbon nanotubes of 1-nm diameter, *Nature,* Vol. 363, No. 6430, pp. 603-605, 1993/06/01, 1993.

[42] A. A. Balandin, S. Ghosh, W. Bao, I. Calizo, D. Teweldebrhan, F. Miao, C. N. Lau, Superior Thermal Conductivity of Single-Layer Graphene, *Nano Letters,* Vol. 8, No. 3, pp. 902-907, 2008/03/01, 2008.

[43] P. Sundqvist, F. J. Garcia-Vidal, F. Flores, M. Moreno-Moreno, C. Gómez-Navarro, J. S. Bunch, J. Gómez-Herrero, Voltage and Length-Dependent Phase Diagram of the Electronic Transport in Carbon Nanotubes, *Nano Letters,* Vol. 7, No. 9, pp. 2568-2573, 2007/09/01, 2007.

[44] J.-H. He, A New Resistance Formulation for Carbon Nanotubes, *Journal of Nanomaterials,* Vol. 2008, No. 1, pp. 954874, 2008.

[45] H. Ha, S. Müller, R.-P. Baumann, B. Hwang, PEAKFORCE QUANTITATIVE NANOMECHANICAL MAPPING FOR SURFACE ENERGY CHARACTERIZATION ON THE NANOSCALE: A MINI-REVIEW, *2024*, pp. 12, 2024-04-01, 2024.

[46] M. A. Ilgamov, A. A. Aitbaeva, I. S. Pavlov, S. V. Dmitriev, CARBON NANOTUBE UNDER PULSED PRESSURE, *2024*, pp. 18, 2024-07-31, 2024.

[47] N. Anjum, J.-H. He, Geometric potential in nano/microelectromechanical systems: Part I mathematical model, *International Journal of Geometric Methods in Modern Physics,* Vol. 0, No. 0, pp. 2440027.

[48] R. Zhu, K. Li, D. Wang, J. Fei, J. Yan Tan, S. Li, J. Zhang, H. Li, S. Fu, Biomimetic optimized concept with Murray networks for accelerated solar-driven water evaporation, *Chemical Engineering Journal,* Vol. 467, pp. 143383, 2023/07/01/, 2023.

[49] D. Ma, G. Wang, Y. Ma, C. Zhu, X. Tang, Hydrodynamics behavior and mass transfer performance of gas–liquid two-phase flow in the honeycomb fractal microreactor, *Chemical Engineering Journal,* Vol. 462, pp. 142228, 2023/04/15/, 2023.

[50] J. Liao, J. Nie, B. Sun, T. Jiao, M. Zhang, S. Song, A cellulose composite filter with multi-stage pores had high filtration efficiency, low pressure drop, and degradable properties, *Chemical Engineering Journal,* Vol. 482, pp. 148908, 2024.