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RESEARCH PAPER

Variational approach to optimal control constrained by fractal-fractional differential equations

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Abstract

The extant corpus of literature pertaining to optimal control problems with partial differential equation (PDE) constraints is extensive. This paper introduces a novel variational approach to optimal control problems constrained by fractal-fractional differential equations. Utilizing the shallow water wave as a case study, the semi-inverse method is employed to establish the variational formulation. This approach not only exemplifies a novel mode of thinking but also has significant ramifications for the field. This novel approach to optimal control paves a promising path for further research and provides researchers and practitioners with a novel perspective and potential avenues for further exploration. By exploring this alternative approach, researchers and practitioners can develop a more profound understanding of the fundamental nature of optimal control problems and identify more effective solutions for a wide range of applications.

Keywords: Fractal-Fractional Differential Equations; Optimal Control Problem; Shallow Water Equations; Control Constraints; Semi-Inverse method; Lagrange multiplier; Variational Methods; Fractional Calculus Applications.

1. Introduction

Optimization with partial differential equation (PDE) constraints is of paramount importance in both the domains of mathematics and engineering [1], Tindano, et al. focused on nonlinear elliptical evolution equations with incomplete data [2], Garg & Rani applied differential equations to optimize maintenance schedules in a coil shop, a critical component of manufacturing systems [3], Zambelongo, et al. integrated optimal control theory to determine harvesting efforts that maximize yield while preventing species collapse [4].

In the field of economics, it plays a crucial role in helping decision-makers make strategic decisions and optimize resource allocation [5]. By applying optimization with PDE constraints, economists can analyze complex economic models and systems, taking into account various factors and constraints. This enables them to develop more effective economic policies and strategies, leading to improved economic outcomes.

In the realm of drug therapy, it can be utilized to design optimal treatment plans [6]. By considering the complex biological processes and constraints described by partial differential equations, researchers and clinicians can develop more personalized and effective treatment plans. This can lead to better patient outcomes and improved quality of life.

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Medical image analysis also benefits significantly from it for enhancing diagnostic accuracy [7]. By incorporating the physical and biological properties of imaging processes into the optimization framework, it is possible to improve the quality and accuracy of medical images. This can lead to more accurate diagnoses and better treatment decisions.

In mechanical engineering, it aids in optimizing designs for better performance [8]. By considering the mechanical properties and constraints described by partial differential equations, engineers can design more efficient and reliable structures and systems. This can lead to improved performance, reduced costs, and increased safety.

Additionally, numerical methods for optimal control problems have been the subject of extensive research [9, 10]. Scholars and researchers have been constantly exploring and developing new numerical techniques to solve optimal control problems with PDE constraints. This research has led to significant advances in the field and has opened up new possibilities for applications. Non-classical optimal control problem [11], nonsmooth optimal control problem [12] and indirect optimization [13] were appeared in various practical applications.

The above research has exemplified the versatility of optimal control theory in addressing interdisciplinary challenges. From industrial maintenance and economic policy to ecological conservation and nonsmooth dynamical systems, the studies highlight innovative methodologies—such as hybrid optimization, data imputation, and ecological modeling—to solve real-world problems. Each contribution underscores the critical role of differential equations and control frameworks in advancing both theoretical and applied research. However, the variational method for the optimal control problem within the shallow water equations in a fractal space remains uncharted territory in the open literature. In this paper, we aim to shed light on its unique advantages, opening up new avenues for research and application.

The variational principle is a fundamental concept that holds great significance in both physics and mathematics. It asserts that the true path or state of a physical system is the one that extremizes a certain functional. This principle serves as a powerful and essential tool for analyzing and understanding a wide variety of phenomena.

In mechanics, for instance, the principle of least action is a variational principle that determines the governing equations. By minimizing the action functional, one can derive the equations of motion for a mechanical system. This principle has been widely used in classical mechanics and has led to many important discoveries and insights.

Variational principles are also employed in fields such as electromagnetism, quantum mechanics, and fluid dynamics. They frequently lead to elegant mathematical formulations and offer insights into the underlying nature of physical systems. In electromagnetism, variational principles are used to derive Maxwell's equations. In quantum mechanics, the variational method is often used to approximate the ground state energy and wave function of a quantum system. In fluid dynamics, variational principles can be used to analyze and understand fluid flow phenomena.

Many new variational formulations have emerged for incompressible flow [14], tsunami waves [15], various shallow waves [16, 17], lubrication [18], MEMS systems [19, 20]. These new formulations have provided new perspectives and methods for analyzing and understanding these complex phenomena. They have also led to new applications and technological developments.

2. Optimal control problems constrained by a variational principle in a fractal space

The variational principle holds tremendous potential for application in optimization problems that are constrained by differential equations. The overarching goal in such scenarios is frequently to substitute the constraints with a meticulously crafted variational formulation. This approach is aimed at transforming the optimization problem into a matter of extremizing a specific functional. By doing so, it becomes possible to utilize powerful mathematical techniques such as the calculus of variations to painstakingly search for the optimal solution.

The variational principle also presents several distinct advantages. For instance, it offers a global perspective on the problem at hand, enabling a more comprehensive understanding. Additionally, it allows for the careful consideration of constraints, which is crucial in many real-world applications. With its ability to handle complex problems involving multiple variables and constraints, it emerges as a highly valuable tool in the realm of optimization.

Optimization in a fractal space represents a truly unique and exceptionally challenging area of study. Fractal spaces possess extremely intricate and self-similar structures, which significantly add to the complexity of the optimization process. In such spaces, traditional optimization methods may not be directly applicable due to the unique characteristics and challenges presented by the fractal nature.

The optimal control problems constrained by a variational principle in a fractal space are designed to thoroughly investigate the variational method for the optimal control problem. This area of research holds great promise for uncovering new approaches and solutions to complex optimization challenges in fractal environments.

Once again, the variational principle has significant potential application to optimization problems constrained by differential equations. The common objective is often to replace the constraints with a precisely formulated variational formulation. By formulating an optimization problem in terms of a variational principle, we can effectively transform it into a problem of extremizing a particular functional. This, in turn, allows us to employ mathematical techniques like the calculus of variations to diligently seek out the most optimal solution.

The variational principle continues to offer a host of advantages. It provides a panoramic view of the problem, facilitating a deeper understanding. Moreover, it enables the incorporation of constraints, which is essential for addressing practical optimization issues. Its capacity to handle complex problems with numerous variables and constraints solidifies its status as a valuable asset in optimization.

Optimization within a fractal space remains a one-of-a-kind and arduous area of study. The fractal spaces boast elaborate and self-similar architectures that greatly enhance the complexity of the optimization task. In these spaces, traditional optimization methods might not be readily applicable, necessitating the exploration of alternative approaches.

The optimal control problems constrained by a variational principle in a fractal space is to investigate the variational method for the optimal control problem:

$$J_1(u,w)=\iint_{\Omega}\left\{\frac{1}{2}(u-u_d)^2+\frac{1}{2}\lambda w^2\right\}dt^{\alpha}dx^{\alpha}\to\min. \tag{1}$$

subject to

$$J_2(u)=\iint_{\Omega}L(u,u_{t^{\alpha}},u_{x^{\alpha}},\cdots,w)dt^{\alpha}dx^{\alpha}\to\min. \tag{2}$$

where J(u,w) is the objective functional, u_d is a desired state， w is the control variable，the subscriptions in Eq.(2) imply the two-scale fractal derivatives [21, 22], α is the two-scale fractal dimensions [23]. The two-scale fractal geometry has been widely used in practical problems, for examples, porous concretes [24, 25], porous beams [26], Clover-Inspired Fractal Architecture [27], nanofiber members [28], fractal super-ropes [29]. The fractal-fractional models are now widely using for dealing with complex problems, e.g., heat transfer [30] and fractal vibration system [31-33].

The main challenge resides in handling the complexity of variational principles and formulating efficient optimization algorithms. Researchers employ the semi-inverse method [34-36] to address the establishment of a variational formulation. By optimizing systems with variational principal constraints, we can enhance performance, design better structures, and solve complex engineering problems. This area of study continues to develop, presenting new opportunities for innovation and improvement.

Minimizing Eq. (2), we can obtain the Euler-Lagrange differential equation

$$\frac{\partial L}{\partial u}-\frac{\partial^{\alpha}}{\partial t^{\alpha}}\left(\frac{\partial L}{\partial u_{t^{\alpha}}}\right)-\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{\partial L}{\partial u_{x^{\alpha}}}\right)+\cdots=0 \tag{3}$$

When $\alpha=1$, optimal control problems constrained by Eq. (3) have been the subject of extensive study in the literature [9, 10].

3. Shallow Water equations and variational formulation

Shallow water waves, as mentioned in [37], have a profound impact on safety along the coastline. In particular, tsunami waves pose a significant potential threat to the biological environment as well as the security of people's lives and properties. These waves are characterized by the fact that their behavior is greatly influenced by the surface morphology of the seabed.

The traditional models for shallow water waves are typically differential equations. However, these equations are often unable to fully describe the effects of an unsmooth seabed. Chinese mathematician Yan Wang and his team have made remarkable discoveries. They have revealed that fractal boundaries play a crucial role in the prevention of tsunami-like waves, as documented in [38, 39]. This finding offers new perspectives and potential solutions for understanding and mitigating the risks associated with these powerful natural phenomena.

Considering the fractal boundary of the seabed, we consider the following shallow water equations:

$$\frac{\partial^\alpha u}{\partial t^\alpha}+au\frac{\partial^\alpha u}{\partial x^\alpha}+b\frac{\partial^\alpha v}{\partial x^\alpha}=0 \tag{4}$$

$$\frac{\partial^\alpha v}{\partial t^\alpha}+m\frac{\partial^\alpha u}{\partial x^\alpha}+n\frac{\partial^\alpha (uv)}{\partial x^\alpha}+p\frac{\partial^{3\alpha} u}{\partial x^{3\alpha}}=0 \tag{5}$$

where *a, b, m, n* and *p* are constants.

The semi-inverse method, as indicated in [34], is an extremely valuable and highly useful approach for the establishment of a variational formulation. This remarkable method offers a systematic and well-structured way to deal with complex problems by effectively reducing the complexity involved in deriving variational principles.

It commences by making an educated and informed guess or assumption about the form of the solution or certain crucial aspects of the problem. This initial educated guess serves as a guiding light and helps to significantly narrow down the extensive search space. Then, by employing a variety of mathematical techniques and fundamental principles, the semi-inverse method proceeds to skillfully manipulate and transform the problem in order to fit it within a variational framework.

This approach can be particularly useful and highly beneficial when dealing with problems that prove to be extremely difficult to solve using traditional methods. It allows for a more intuitive and flexible way of finding solutions, enabling researchers and practitioners to explore different possibilities and perspectives. The semi-inverse method has found extensive applications in various diverse fields such as mechanics, physics, and engineering. In these fields, it has played a crucial role in helping to solve complex problems related to partial differential equations and optimization. Overall, it provides a powerful and essential tool for establishing variational formulations and gaining a deeper understanding of complex systems.

The semi-inverse method, as mentioned in [34], is extremely applicable to the establishment of variational formulations for Shallow water equations, as demonstrated in [40-42]. It can also be utilized for finding a variational principle for MEMS, as indicated in [43].

Following a similar way as that in Ref. [34] `, we re-write Eqs. (5) and (6) in the form

$$\frac{\partial^\alpha u}{\partial t^\alpha}+\frac{\partial^\alpha}{\partial x^\alpha}(\frac{a}{2}u^2+bv)=0 \tag{6}$$

$$\frac{\partial^\alpha v}{\partial t^\alpha}+\frac{\partial^\alpha u}{\partial x^\alpha}(mu+nuv+pu_{xx})=0 \tag{7}$$

We can introduce two functions φ and ψ defined, respectively, as

$$\begin{cases}\dfrac{\partial^\alpha \varphi}{\partial x^\alpha}=u\\[2mm] \dfrac{\partial^\alpha \varphi}{\partial t^\alpha}=-(\dfrac{a}{2}u^2+bv)\end{cases} \tag{8}$$

and

$$\begin{cases}\dfrac{\partial^\alpha \psi}{\partial x^\alpha}=v\\[2mm] \dfrac{\partial^\alpha \psi}{\partial t^\alpha}=-(mu+nuv+p\dfrac{\partial^{2\alpha} u}{\partial x^{2\alpha}})\end{cases} \tag{9}$$

According to the semi-inverse method [34], we construct a trial-variational formulation in the form

$$J_3(u,v,\varphi)=\iint_{\Omega}\left\{v\frac{\partial^{\alpha}\varphi}{\partial t^{\alpha}}+(mu+nuv+p\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}})\frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}+F\right\}dt^{\alpha}dx^{\alpha} \tag{10}$$

where F is an unknown function.
The Euler-Lagrange equations of Eq. (10) with respect to u and v are, respectively, as

$$m\frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}+nv\frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}+p\frac{\partial^{3\alpha}\varphi}{\partial x^{3\alpha}}+F_u=0 \tag{11}$$

$$\frac{\partial^{\alpha}\varphi}{\partial t^{\alpha}}+nu\frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}+F_v=0 \tag{12}$$

where F_u and F_v are variational derivatives with respect to u and v, respectively. In view of Eq. (8), we have

$$F_u=-m\frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}-nv\frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}-p\frac{\partial^{3\alpha}\varphi}{\partial x^{3\alpha}}=-mu-nuv-p\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}} \tag{13}$$

$$F_v=-\frac{\partial^{\alpha}\varphi}{\partial t^{\alpha}}-nu\frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}=\frac{a}{2}u^2+bv-nu^2 \tag{14}$$

In order to identify *F* from Eqs.(13) and (14), it requires that

$$\frac{\partial}{\partial v}(F_u)=\frac{\partial}{\partial u}(F_v) \tag{15}$$

This results in

$$-nu=(a-2n)u \tag{16}$$

Eq. (16) implies

$$n=a \tag{17}$$

Under the requirement of Eq. (17), *F* can be determined as

$$F=-\frac{1}{2}mu^2-\frac{1}{2}nu^2v+\frac{1}{2}p(\frac{\partial^{\alpha}u}{\partial x^{\alpha}})^2+\frac{1}{2}bv^2 \tag{18}$$

So we obtain the following variational formulation

$$J_3(u,v,\varphi)=\iint_\Omega\left\{v\frac{\partial^\alpha\varphi}{\partial t^\alpha}+(mu+nuv+p\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}})\frac{\partial^\alpha\varphi}{\partial x^\alpha}\right\}dt^\alpha dx^\alpha$$
$$+\iint_\Omega\left\{-\frac{1}{2}mu^2-\frac{1}{2}nu^2v+\frac{1}{2}p(\frac{\partial^\alpha u}{\partial x^\alpha})^2+\frac{1}{2}bv^2\right\}dt^\alpha dx^\alpha \tag{19}$$

and the following constrained variational principle

$$J_4(\varphi)=\iint_\Omega\left\{-\frac{1}{2}mu^2-\frac{1}{2}nu^2v+\frac{1}{2}p(\frac{\partial^\alpha u}{\partial x^\alpha})^2+\frac{1}{2}bv^2\right\}dt^\alpha dx^\alpha \tag{20}$$

The variational principle of Eq. (30) is under constraints of Eq. (8).

4. Variational method for optimal control problems

The variational method for the optimal control problem now can be expressed as

$$J_5(\varphi,w)=\iint_\Omega\left\{\frac{1}{2}(\varphi-\varphi_d)^2+\frac{1}{2}\lambda w^2\right\}dt^\alpha dx^\alpha\to\min. \tag{21}$$

s.t

$$\begin{cases}J_6(\varphi,w)=\iint_\Omega\left\{-\frac{1}{2}mu^2-\frac{1}{2}nu^2v+\frac{1}{2}p(\frac{\partial^\alpha u}{\partial x^\alpha})^2+\frac{1}{2}bv^2+w\varphi\right\}dt^\alpha dx^\alpha\\ \text{s.t.}\frac{\partial^\alpha\varphi}{\partial x^\alpha}=u,\frac{\partial^\alpha\varphi}{\partial t^\alpha}=-(\frac{a}{2}u^2+bv)\end{cases} \tag{22}$$

This type of optimization is truly groundbreaking as it has not been seen in the literature before. It stands out for its mathematical rigor, ensuring a solid theoretical foundation. The approach is not only mathematically sound but also provides physical insights, helping to better understand the underlying phenomena. Moreover, it is numerically simple, making it accessible and practical for implementation. This combination of qualities makes it a valuable addition to the field of optimization. It has the potential to open up new avenues of research and application, inspiring further exploration and innovation. With its unique characteristics, it is poised to make a significant impact on various disciplines that rely on optimization techniques.

The above optimization has still constraints of differential equations, and the following optimization can be easily obtained:

$$J_6(u,v,\varphi,w)=\iint_\Omega\left\{\frac{1}{2}\left\{u-u_d)^2+(v-v_d)^2+(\varphi-\varphi_d)^2\right\}\right\}dt^\alpha dx^\alpha$$
$$+\iint_\Omega\left\{\frac{1}{2}\left\{\lambda_1(w_1)^2+\lambda_2(w_2)^2+\lambda_3(w_3)^2\right\}\right\}dt^\alpha dx^\alpha\to\min. \tag{23}$$

s.t.

$$J_7(u,v,\varphi)=\iint_{\Omega}\left\{v\frac{\partial^{\alpha}\varphi}{\partial t^{\alpha}}+(mu+nuv+p\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}})\frac{\partial^{\alpha}\varphi}{\partial x^{\alpha}}\right\}dt^{\alpha}dx^{\alpha}$$
$$+\iint_{\Omega}\left\{-\frac{1}{2}mu^2-\frac{1}{2}nu^2v+\frac{1}{2}p(\frac{\partial^{\alpha}u}{\partial x^{\alpha}})^2+\frac{1}{2}bv^2\right\}dt^{\alpha}dx^{\alpha}$$
$$+\iint_{\Omega}\{uw_1+vw_1+\phi w_3\}dt^{\alpha}dx^{\alpha} \tag{24}$$

This optimization eliminates the constrains with differential equations, now it is constructed an optimization using Lagrange multiplier in a single object function:

$$J_8(u,v,\varphi,w)=J_6(u,v,\varphi,w)+\lambda J_7(u,v,\varphi) \tag{25}$$

where λ is the Lagrange multiplier.

5. Discussion and Conclusion

The variational principle can be effectively utilized to solve optimization problems with partial differential equation (PDE) constraints in the following manner: First and foremost, by formulating a meticulous variational formulation for the partial differential equation, the overarching goal is frequently to replace the PDE constraint with a precisely crafted variational formulation. By doing so, it becomes possible to apply the variational-based analytical methods and numerical methods. This enables the determination of the optimal solution by painstakingly finding the path or state that extremizes the functional.

The variational principle offers several distinct advantages in this context. It provides a comprehensive global perspective on the problem at hand. This means that it can consider the overall behavior of the system rather than being confined to just local aspects. Additionally, it allows for the effortless consideration of constraints, as these can be directly incorporated into the variational formulation.

For problems that are constrained by PDEs, the variational principle can adeptly handle complex systems with multiple variables and constraints. This makes it an extremely valuable tool in the realm of optimization, as it can effectively deal with the complexity introduced by the partial differential equations.

In conclusion, this paper's in-depth exploration of optimal control problems constrained by a variational formulation truly represents a remarkable breakthrough. By using the shallow water wave as an illustrative example and employing the semi-inverse method, it has opened up a brand-new door to the world of optimal control. This novel approach not only offers researchers and practitioners a fresh perspective but also holds the promise of uncovering more efficient solutions for a wide range of applications.

What makes this work even more significant is that this particular type of optimization has not been witnessed in the literature before. It fills a crucial gap and paves the way for further research and innovation. As we continue to delve deeper into this alternative path, we can look forward to gaining deeper insights into the nature of optimal control problems. A future filled with innovative solutions that will reshape the field is on the horizon. This discovery has the potential to inspire new lines of inquiry and open up unexplored territories in both the theoretical and practical aspects of optimization. It holds the promise of driving progress and advancing our understanding of complex systems and optimization techniques.

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